The matrix product ansatz for the six-vertex model

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Abstract

Recently it was shown that the eigenfunctions for the the asymmetric exclusion problem and several of its generalizations as well as a huge family of quantum chains, like the anisotropic Heisenberg model, Fateev-Zamolodchikov model, Izergin-Korepin model, Sutherland model, $t-J$ model, Hubbard model, etc, can be expressed by a matrix product *ansatz*. Differently from the coordinate Bethe *ansatz*, where the eigenvalues and eigenvectors are plane wave combinations, in this *ansatz* the components of the eigenfunctions are obtained through the algebraic properties of properly defined matrices. In this work, we introduce a formulation of a matrix product *ansatz* for the six-vertex model with periodic boundary condition, which is the paradigmatic example of integrability in two dimensions. Remarkably, our studies of the six-vertex model are in agreement with the conjecture that all models exactly solved by the Bethe *ansatz* can also be solved by an appropriated matrix product *ansatz*.

1 Introduction

The Bethe *ansatz* in its several formulations (coordinate, inverse scattering and functional) has been established over the years as a powerful tool for the description of the eigenvectors of a huge variety of integrable one-dimensional quantum spin chains and two-dimensional transfer matrices (see e.g. [1]-[5] for reviews). A quantum Hamiltonian or a transfer matrix is considered exactly integrable if an infinite number of its eigenstates can be expressed by the Bethe *ansatz* in the thermodynamic limit. In the last two decades it has been shown that a matrix product *ansatz* (MPA) can be used to express the stationary distribution of the probability densities of some special stochastic models [6]-[9]. Although these models are in general not integrable through the Bethe *ansatz*, they have the components of its ground-state wavefunctions given in terms of a product of matrices. According to this *ansatz*, the algebraic properties of the matrices defining the MPA fix the components of the wavefunction apart from a normalization constant.

An important development of the MPA that appeared in the context of stochastic models is the dynamical matrix product *ansatz* (DMPA) [10, 11]. This DMPA was shown originally to be valid to the problem of asymmetric diffusion of particles on the lattice [10] and extended to other stochastic models and related spin Hamiltonians [12, 13]. This *ansatz* allows the calculation of the probability densities, of the stochastic system, at arbitrary times. In the related spin Hamiltonian this DMPA asserts that not only the ground-state wave function, as in the standard MPA, but an arbitrary wavefunction have its components expressed in terms of a matrix product *ansatz* whose matrices, in distinction of the standard MPA, are now time dependent.

More recently [14]-[16] it was shown that several exactly solvable Hamiltonians, related or not to stochastic models, may also be solvable by an appropriate time independent matrix product *ansatz*. In this new MPA not only the ground-state but all wavefunctions can be expressed by a product of matrices. Using this new MPA it was possible to rederivethe results previously obtained through the Bethe *ansatz* for several quantum chains with one and two global conservation laws, such as the XXZ chain, spin-1 Fateev-Zamolodchikov model, Izergin-Korepin model, Sutherland model, t-J model, Hubbard model, etc [14, 15], as well as the exact solution of the asymmetric exclusion problem with particles of arbitrary size [16]. Moreover, the components of the eigenfunctions of the exact integrable Hamiltonians, which according to the Bethe *ansatz* are normally given by a combination of plane waves, can also be obtained from the algebraic properties of the matrices defining the new MPA. In the case of Bethe *ansatz* solutions the eigenvalues and the amplitudes of the plane waves are fixed apart from a normalization constant by the eigenvalue equation of the Hamiltonian. On the other hand, in the new MPA, the eigenvalue equation fix the commutation relations of the matrices defining the *ansatz*. The advantage of the new MPA in the search for new exact integrable models, as showed in our previous works [14]-[16], is its simplicity and unifying character in the implementation for arbitrary systems. All the previous successful applications of the new MPA [14]-[16] was concerned with the eigenspectrum of quantum Hamiltonians and stochastic models. In this paper we are going to show that these results can also be extended to transfer matrix calculations of two-dimensional classical spin models. More specifically we are going to extend the new MPA introduced in [14]-[16] for the case of the row-to-row transfer matrix of the six-vertex model with toroidal boundary condition, which is the paradigm of integrability in two dimensions. This transfer matrix was diagonalized through the coordinate Bethe *ansatz* firstly by Lieb [17], in a special case, and by Sutherland [18] and Yang [19, 20].

2 The asymmetric six-vertex model and its transfer matrix

The six-vertex model defined on a square lattice, was introduced to explain the residual entropy of the ice [17]-[20]. We are going to consider the asymmetric version of the six-vertex model that was first studied and exactly solved with standard methods [21, 22]. This model is defined on a square lattice with M rows and N columns and toroidal boundary condition. At each horizontal (vertical) lattice bond we attach an arrow pointing to the left or right (up or down) directions. These arrows configurations can be equivalently described by the vertex configurations of the lattice. A vertex configuration at a given site (center) is formed by the four arrows attached to its links. The allowed vertex configurations are those satisfying the ice rules: two of the arrows pointing inward and the other two pointing outward of its center. There are six possible configurations for the vertices. Theses configurations are showed in fig. 1a. In fig. 1b, a more convenient notation is introduced, in which we only draw by a solid line (broken line) the links having arrows pointing to the left or down (right or up) of the center defining the vertex. Labeling the M rows sequentially by $m = 1, 2, ..., M$ and by $\{x^m\}$ the solid line positions on the vertical edges of the row m, the partition function can be written as

$$Z = \sum_{\{x^1\}}\sum_{\{x^2\}}\cdots\sum_{\{x^M\}} T(\{x^1\},\{x^2\})T(\{x^2\},\{x^3\})\cdots T(\{x^M\},\{x^1\}) = \mathrm{Tr}(T^M), \tag{1}$$

where T is the $2^N \times 2^N$ transfer matrix, with elements

$$T(\{y\},\{x\}) = \sum e^{-\beta(n_1\epsilon_1+n_2\epsilon_2+\cdots+n_6\epsilon_6)}, \tag{2}$$

where the summation is over all allowed arrangements of lines on the horizontal edges and n_j $(j = 1, ..., 6)$ are the numbers of vertices of types $(1, ..., 6)$ formed by the configurations. For convenience we label the Boltzmann weights $a_0, a_1, b_1, b_2, c_1, c_2$ associated with the vertices as in fig. 1.

It is also important to mention that the number of vertical and horizontal lines is conserved, forming continuous non-crossing paths through the lattice. On the other hand the transfer matrix, due to the toroidal boundary condition, is translation invariant. As a consequence of these symmetries the transfer matrix breaks up into blocks of disjoint sectors labeled by the number n of vertical lines ($n = 0, ..., N$) and the momentum eigenvalues P ($P = \frac{2\pi}{N} l, l = 0, 1, ..., N-1$).

3 The Matrix Product *ansatz* for the six-vertex model

The *ansatz* we propose [14]-[16] states that any eigenfunction $|\Psi_{n,P}\rangle$ of the transfer matrix (2) in the sector with n ($n = 0, 1, 2, \ldots, N$) vertical lines and momentum P ($P = \frac{2\pi}{N} l, l = 0, 1, ..., N-1$) is given in terms of a matrix product, i. e., their amplitudes are given by the trace of the following matrix product:

$$|\psi_{n,P}\rangle = \sum_{x_1,\ldots,x_n}^{*} \mathrm{Tr}(E^{x_1-1} A E^{x_2-x_1-1} A \cdots E^{x_n-x_{n-1}-1} A E^{L-x_n} \Omega_P) |x_1, \ldots, x_n\rangle, \tag{3}$$

where $|x_1, \ldots, x_n\rangle$ denote the configurations with vertical lines at positions $(x_1, \ldots, x_n)$ and the symbol $(*)$ in the sum means the restriction to the configurations where $L \geq x_{i+1} > x_i \geq 1$. The objects A, E and Ω_P are abstract matrices, or operators, with an associative product whose commutation relations will be fixed by imposing the validity of the eigenvalue equation of the transfer matrix (2). The matrices A and E are associated with the sites where we have a vertical line or not, respectively, and the matrix Ω_P is introduced in order to fix the momentum P of the eigenfunction $|\Psi_{n,P}\rangle$. The fact that $|\psi_{n,P}\rangle$ has a momentum P imply that the ratio of the amplitudes corresponding to the configurations $|x_1, \ldots, x_n\rangle$ and $|x_1+1, \ldots, x_n+1\rangle$ is e^{-iP}, i.e.,

$$\frac{\mathrm{Tr}(E^{x_1-1} A E^{x_2-x_1-1} A \cdots E^{x_n-x_{n-1}-1} A E^{L-x_n} \Omega_P)}{\mathrm{Tr}(E^{x_1} A E^{x_2-x_1-1} A \cdots E^{x_n-x_{n-1}-1} A E^{L-x_n-1} \Omega_P)} = e^{-iP}, \tag{4}$$

and consequently from (3) we obtain the following commutation relations

$$A\Omega_P = e^{-iP} \Omega_P A, \;\; E\Omega_P = e^{-iP} \Omega_P E. \tag{5}$$

The matrix product *ansatz* will be valid if the algebraic relations among the matrices A, E and Ω_P are consistent with the constrains imposed by the eigenvalue equation

$$T|\psi_{n,P}\rangle = \Lambda_n|\psi_{n,P}\rangle. \tag{6}$$

To solve this last equation, it is helpful to begin, as usual, by considering the simple cases where $n = 0, 1$ and 2 before considering the general case.

The case n = 0.

In this case the solution of the eigenvalue equation (6) is trivial since we do not have vertical lines between two successive rows. There are only two possible horizontal arrangements either all bonds have a line or all of them are empty. In this case the vertices are all of type 1 or type 3 (see figure 1) and consequently the eigenvalue is given by

$$\Lambda_0 = a_0^N + b_1^N \tag{7}$$

where a_0 and b_1 are the Boltzmann weights of the vertices of types 1 and 3, respectively (see figure 1).

The case n = 1.

We have in this case just one vertical line between two rows. The transfer matrix links a vertical line at position y ($y = 1, .., N$) above a row to a vertical line at any position x ($x = 1, ..., N$) under this row. The elements of the transfer matrix $T(y, x)$ in this sector with momentum P are given by (2). They are the product of the Boltzmann weights of the vertex appearing on the row. If the position of the line x is less (greater) than y, the vertex configuration at these sites will be of types 5 and 6 (6 and 5) and all the others vertices will be of types 3 (1) and 1 (3) depending on whether the vertices are between the positions x and y, or not, respectively. In the case where $x = y$ these vertices should be of type 4 or 2 with all the remains vertices of type 1 or 3 respectively. Consequently the eigenvalue equation (6) for the transfer matrix (2) associated with the components of $|\psi_{n,P}\rangle$ (3) with $n = 1$ and momentum P give us the relations

$$\begin{aligned}\Lambda_1 \mathrm{Tr}(E^{x-1}AE^{N-x}\Omega_P) = \quad & \sum_{y=x+1}^{N} a_0^{N-y+x-1} b_1^{y-x-1} c_1 c_2 Tr(E^{y-1}AE^{N-y}\Omega_P) + \\ & \sum_{y=1}^{x-1} a_0^{x-y-1} b_1^{N-x+y-1} c_1 c_2 Tr(E^{y-1}AE^{N-y}\Omega_P) + \\ & (a_0^{N-1}b_2 + b_1^{N-1}a_1)\mathrm{Tr}(E^{x-1}AE^{N-x}\Omega_P). \end{aligned} \tag{8}$$

Equation (8) can be simplified in order to express all the matrix products in terms of a single one. This is possible by exploring the cyclic property of the trace as well as the commutation relations (5). This allow us to factorize the matrix product in the following form

$$\Lambda_1 = \sum_{y=x+1}^{N} a_0^{N-y+x-1} b_1^{y-x-1} c_1 c_2 e^{-iP(y-x)} + \sum_{y=1}^{x-1} a_0^{x-y-1} b_1^{N-x+y-1} c_1 c_2 e^{-iP(y-x)} + (a_0^{N-1} b_2 + b_1^{N-1} a_1). \quad (9)$$

By evaluating the sums in (9) we obtain

$$\Lambda_1 = a_0^N L(P) + b_1^N M(P) + b_1^N \frac{c_1 c_2}{a_0} \left(\frac{b_1}{a_0}\right)^{-x} \frac{e^{-iP(1-x)}}{a_0 - b_1 e^{-iP}} (1 - e^{-iNP}), \quad (10)$$

where

$$L(P) = \frac{a_0 b_2 + (c_1 c_2 - b_1 b_2) e^{-iP}}{a_0^2 - a_0 b_1 e^{-iP}} \quad \text{and} \quad M(P) = \frac{a_0 a_1 - c_1 c_2 - a_1 b_1 e^{-iP}}{a_0 b_1 - b_1^2 e^{-iP}}. \quad (11)$$

In order to satisfy (6), the eigenvalue Λ_1 in (10) should be independent of the vertical line position x. Thus the last term in the right hand side of (10) must vanish. The only way to cancel this term, for non zero Boltzmann weights, is obtained by imposing the following constraint to the momentum P

$$e^{iNP} = 1, \quad (12)$$

which is automatically satisfied, since $P = \frac{2\pi}{N} l \; l = 0, 1, ..., N-1$. The eigenvalue (10) is then given by

$$\Lambda_1 = a_0^N L(P) + b_1^N M(P). \quad (13)$$

An alternative solution of (8), whose generalization will be convenient for arbitrary values of n, is obtained by expressing the matrix A in terms of the matrix E and a spectral parameter dependent matrix

$$A = A_k E, \quad (14)$$

with A_k satisfying

$$E A_k = e^{ik} A_k E. \quad (15)$$

As a consequence of (5) and (15) A_k also satisfies

$$A_k \Omega_P = \Omega_P A_k. \quad (16)$$

The spectral parameter k will be fixed by the eigenvalue equation (8). Inserting (14) in (8) and using the commutation relation (15) we obtain (9) with the value k replacing P. Therefore

$$\Lambda_1 = a_0^N L(k) + b_1^N M(k), \quad (17)$$

with

$$e^{iNk}=1, \qquad k=\frac{2\pi}{N}l \quad (l=0,1,...,N-1). \tag{18}$$

Comparing (12) and (13) with (17) and (18) we observe the equality $k=P$. This fact can also be seen directly by inserting (14) in (4) and using (15).

We still need to verify whether the algebraic relations among the matrices A_k, E and Ω_P (5), (15) and (16) are consistent with the cyclic property of the trace. Indeed these equations yield

$$\begin{aligned}\mathrm{Tr}(A_kE^N\Omega_P) \quad &= e^{-iNk}\mathrm{Tr}(E^NA_k\Omega_P)=e^{-iNk}\mathrm{Tr}(E^N\Omega_PA_k)\\ &= e^{-iNk}\mathrm{Tr}(A_kE^N\Omega_P),\end{aligned} \tag{19}$$

which satisfies the cyclicity of the trace due to (18). Since no new constraints is obtained for the matrices A_k, E and Ω_P, with $k=P$, and for spectral parameter k, the MPA is consistent.

The case n = 2.

In this sector there are two vertical lines in the row. We have in general two types of relations, which are relations where at least one of the vertical lines (y_1,y_2) coincide with (x_1,x_2) and those where y_1 and y_2 interlace with x_1 and x_2 ($x_1<y_1<x_2<y_2$ or $y_1<x_1<y_2<x_2$). Then, the eigenvalue equation (6) imply

$$\begin{aligned}\Lambda_2 \quad & \mathrm{Tr}(E^{x_1-1}AE^{x_2-x_1-1}AE^{N-x_2}\Omega_P)=\\ & \sum_{y_1=x_1}^{x_2}\sum_{y_2=x_2}^{N}{}^{*}\, a_0^{N-y_2+x_1-1}c_2f(x_1,y_1)g(y_1,x_2)f(x_2,y_2)\mathrm{T}r(E^{y_1-1}AE^{y_2-y_1-1}AE^{N-y_2}\Omega_P)+\\ & \sum_{y_1=1}^{x_1}\sum_{y_2=x_1}^{x_2}{}^{*}\, b_1^{N-x_2+y_1-1}c_1g(y_1,x_1)f(x_1,y_2)g(y_2,x_2)\mathrm{T}r(E^{y_1-1}AE^{y_2-y_1-1}AE^{N-y_2}\Omega_P)\end{aligned} \tag{20}$$

where the symbol $*$ in the sums means that terms with $y_1=y_2$ are excluded and

$$f(x,y)=\left\{\begin{array}{ll}\frac{b_2}{c_2} & \text{if } x=y\\ c_1b_1^{y-x-1} & \text{if } y>x\end{array}\right. \quad \text{and} \quad g(y,x)=\left\{\begin{array}{ll}\frac{a_1}{c_1} & \text{if } x=y\\ c_2a_0^{x-y-1} & \text{if } x>y\end{array}\right. . \tag{21}$$

The relation (20) connects configurations where the arrangements of vertical lines above one row do not have the same distance of the vertical lines below this same row. In other words, the distance of the incoming lines y_2-y_1 are in general different of the outcoming distance x_2-x_1. As a consequence, it is not possible to solve the eigenvalue equation by just using the cyclic property of the trace in (3) as done previously in the case $n=1$. We need now to use a generalization of the algebraic relation (14) for the case of two lines. The generalization of (14) is done by writing the matrix A in terms of two new spectral parameter matrices as:

$$A = \sum_{j=1}^{2} A_{k_j} E, \tag{22}$$

with the commutation relations

$$EA_{k_j} = e^{ik_j} A_{k_j} E \quad \text{and} \quad A_{k_j}\Omega_P = \Omega_P A_{k_j} \quad (j = 1, 2), \tag{23}$$

where the spectral parameters k_1 and k_2 are up to now unknown complex numbers.

Inserting (22) in (20) and using in this expression (23) and (11) we obtain, after similar manipulation as we did in the case $n = 1$, the following constraints

$$\begin{aligned}
&\sum_{j,l=1}^{2} \left[\Lambda_2 - a_0^N L(k_j)L(k_l) - b_1^N M(k_j)M(k_l)\right] e^{-ik_jx_1} e^{-ik_lx_2} \mathrm{Tr}(A_{k_j}A_{k_l}E^N\Omega_P) \\
&- \sum_{j,l=1}^{2} a_0^N \left[L(k_l)M(k_j) - \frac{a_1b_2}{a_0b_1}\right] \left(\frac{b_1}{a_0}\right)^{x_2-x_1} e^{-i(k_j+k_l)x_2} \mathrm{Tr}(A_{k_j}A_{k_l}E^N\Omega_P) \\
&- \sum_{j,l=1}^{2} b_1^N \left[L(k_l)M(k_j) - \frac{a_1b_2}{a_0b_1}\right] \left(\frac{b_1}{a_0}\right)^{x_1-x_2} e^{-i(k_j+k_l)x_1} \mathrm{Tr}(A_{k_j}A_{k_l}E^N\Omega_P) \\
&+ \sum_{j,l=1}^{2} b_1^N \frac{c_1^2c_2^2\left[e^{-iNk_l}e^{-ik_jx_1} - e^{-ik_lx_1}\right]e^{-i(k_j+k_l)}}{a_0^2(a_0 - b_1e^{-ik_j})(a_0 - b_1e^{-ik_l})} \left(\frac{b_1}{a_0}\right)^{-x_2} \mathrm{Tr}(A_{k_j}A_{k_l}E^N\Omega_P) \\
&- \sum_{j,l=1}^{2} b_1^N \frac{c_1^2c_2^2\left[e^{-i(N+1)k_l}e^{-ik_jx_2} - e^{-ik_j}e^{-ik_lx_2}\right]}{a_0b_1(a_0 - b_1e^{-ik_j})(a_0 - b_1e^{-ik_l})} \left(\frac{b_1}{a_0}\right)^{-x_1} \mathrm{Tr}(A_{k_j}A_{k_l}E^N\Omega_P) \\
&+ \sum_{j,l=1}^{2} b_1^N c_1c_2b_2 \left[\frac{e^{-i(N+1)k_l}e^{-ik_jx_1}}{a_0^2(a_0 - b_1e^{-ik_l})} - \frac{-e^{-ik_j}e^{-ik_lx_1}}{a_0^2(a_0 - b_1e^{-ik_j})}\right] \left(\frac{b_1}{a_0}\right)^{-x_2} \mathrm{Tr}(A_{k_j}A_{k_l}E^N\Omega_P) \\
&+ \sum_{j,l=1}^{2} b_1^N c_1c_2a_1 \left[\frac{e^{-i(N+1)k_l}e^{-ik_jx_2}}{a_0b_1(a_0 - b_1e^{-ik_l})} - \frac{-e^{-ik_j}e^{-ik_lx_2}}{a_0b_1(a_0 - b_1e^{-ik_j})}\right] \left(\frac{b_1}{a_0}\right)^{-x_1} \mathrm{Tr}(A_{k_j}A_{k_l}E^N\Omega_P) = 0,
\end{aligned} \tag{24}$$

where $1 \le x_1 < x_2 \le N$. This can only be satisfied if each sum is identically zero. Moreover since Λ_2 should be independent of x_1 or x_2 a possible solution of (24) is obtained by imposing

$$\Lambda_2 = a_0^N L(k_1)L(k_2) + b_1^N M(k_1)M(k_2). \tag{25}$$

The algebraic relation between the matrices A_{k_1} and A_{k_2} are obtained by imposing that both the second and third terms in (24) are zero independently, i. e.,

$$A_{k_j}A_{k_l} = -S(k_j,k_l)A_{k_l}A_{k_j} \quad (l \neq j) \quad \left(A_{k_j}\right)^2 = 0 \quad (j,l = 1,2), \tag{26}$$

where

$$S(k_j,k_l) = \frac{L(k_j)M(k_l) - \frac{a_1b_2}{a_0b_1}}{L(k_l)M(k_j) - \frac{a_1b_2}{a_0b_1}}, \tag{27}$$

with $L(k)$ and $M(k)$ given by (11). Finally, the vanishing of the last four terms in (24) will give us relations that fix the spectral parameters values k_1 and k_2. These equations are obtained by exploring the algebraic relations (26)

$$e^{iNk_l} = -S(k_j,k_l) \quad (l,j = 1,2 \text{ and } l \neq j). \tag{28}$$

The eigenvalues and eigenvectors are obtained by inserting the solutions (k_1,k_2) of these last equations in (25) and (27), respectively. The momentum P is obtained by using (22) and (23) in (4), i.e., $P = k_1 + k_2$.

The consistency of the algebraic equations (22), (23) and (27) with the cyclic property of the trace in (3), as in the case $n = 1$, can be easily verified, yielding

$$\begin{aligned} \mathrm{Tr}(A_{k_j}A_{k_l}E^N\Omega_P) &= -S(k_j,k_l)\mathrm{Tr}(A_{k_l}A_{k_j}E^N\Omega_P) \\ &= -S(k_j,k_l)e^{-iNk_j}\mathrm{Tr}(A_{k_l}E^NA_{k_j}\Omega_P) \\ &= -S(k_j,k_l)e^{-iNk_j}\mathrm{Tr}(A_{k_j}A_{k_l}E^N\Omega_P). \end{aligned} \tag{29}$$

The case of general n.

The previous calculation can be extended for arbitrary values of the number n of vertical lines. The transfer matrix (2) when applied to the amplitudes of $|\psi_{n,P}\rangle$ give us an eigenvalue equation linking an arrangement of vertical lines $x_1,...,x_n$ with arrangements $y_1,...,y_n$ with $x_1 \leq y_1 \leq x_2 \leq \cdots x_n \leq y_n$ and $y_1 \leq x_1 \leq y_2 \leq \cdots y_n \leq x_n$. To solve this eigenvalue equation we need to extend the definition (22) and the commutation relations (23) for general n, i. e.,

$$A = \sum_{j=1}^{n} A_{k_j}E \tag{30}$$

with

$$EA_{k_j} = e^{ik_j}A_{k_j}E \quad \text{and} \quad A_{k_j}\Omega_P = \Omega_P A_{k_j} \quad (j = 1,...,n), \tag{31}$$

where k_j $(j = 1,\ldots,n)$ are in general unknown complex numbers that will be fixed by the eigenvalue equation (6). Actually the definition (30) is not the only possible one. The most general definition that enable us to solve the eigenvalue equation is $A = \sum_{j=1}^{n} E^{\alpha} A_{k_j} E^{1+\beta}$, where α and β are integer numbers. However (30) is more convenient since otherwise the S-matrix in (32) and the Bethe equation (33) will depend on the parameters α and β. Inserting (30) in the eigenvalue equation (6) and using the commutation relations (31) we obtain, similarly as done in the case $n = 2$, the algebraic relations among the matrices $\{A_{k_j}\}$

$$A_{k_j} A_{k_l} = -S(k_j, k_l) A_{k_l} A_{k_j} \qquad \left(A_{k_j}\right)^2 = 0 \qquad (j \neq l = 1, ..., n), \tag{32}$$

where $S(k_j, k_l)$ is given by (27) and the spectral parameters k_j $(j = 1, ..., n)$ are fixed by the equation

$$e^{iNk_l} = (-1)^{n-1} \prod_{l=1 \ (l \neq j)}^{n} S(k_j, k_l) \qquad (j = 1, ..., n). \tag{33}$$

No new algebraic relations appear for the matrices $\{A_{k_j}\}$ and the associativity of the algebra (31) and (32) follows from the property $S(k_j, k_l) S(k_l, k_j) = 1$. The eigenvalues for the transfer matrix (2) in the sector with general n is then given by

$$\Lambda_2 = a_0^N L(k_1) L(k_2) \cdots L(k_n) + b_1^N M(k_1) M(k_2) \cdots M(k_n), \tag{34}$$

where $L(k)$ and $M(k)$ are given by (11) and the spectral parameters $\{k_j\}$ are the solutions of (33). The eigenvalues (34) and the spectral parameter equations coincide with the corresponding equations obtained through the Bethe *ansatz* [19, 20].

Finally, the momentum P follows from (4) and (31):

$$P = \sum_{j=1}^{n} k_j. \tag{35}$$

The consistency of the algebraic relations of the matrices defining the MPA (3) with the cyclic property of the trace in (3) is promptly verified as in the cases where $n = 1$ and $n = 2$. Therefore the MPA is consistent and a infinite number of eigenvectors of the transfer matrix (2) can be written by (3) in the thermodynamic limit.

4 Conclusion

In conclusion, we have shown that the new MPA introduced in [14, 15] for one dimensional quantum spin chains, such as the XXZ chain, spin-1 Fateev-Zamolodchikov model, Izergin-Korepin model, Sutherland model, t-J model, Hubbard model, as well as the exact solution of the asymmetric exclusion problem with particles of arbitrary size [16], can also be extended to the diagonalization of the row-to-row transfer matrix of the six-vertex model with toroidal boundary condition. Differently from the standart MPA [6]-[9] this new MPA [14, 15] asserts that all wavefunctions can be expressed by a product of matrices. The solution of the six vertex model through the new MPA is in agreement with the conjecture proposed in [10] and [14, 15] that all models exactly solved by Bethe *ansatz* can also be solved by an appropriate MPA. An interesting problem for the future is the formulation of a MPA for others spin models like the 8-vertex model, which is related to quantum spin chains with no global conservation laws such as the XYZ chain.

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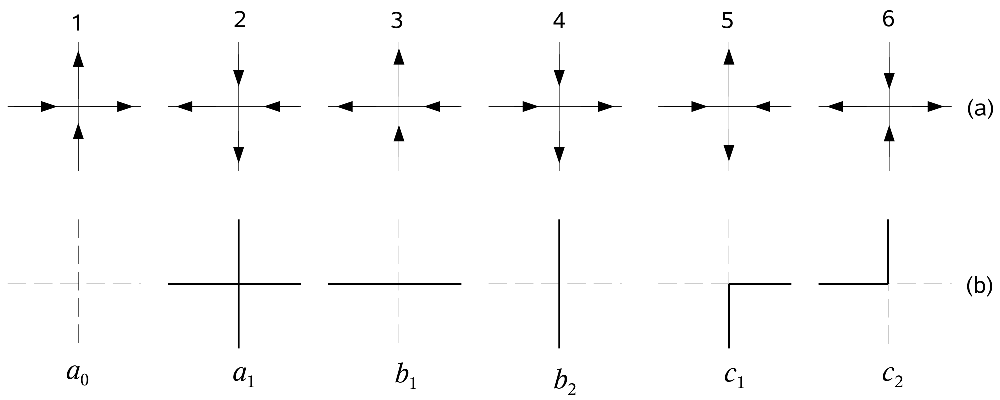

Figure 1: The six vertex configurations and their related Boltzmann weights. In (a) we draw all the arrows and in (b) we draw by solid lines th links where the arrows are pointing to the down and left directions.