# Relativistic particle dynamics in $D = 2 + 1$.

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**Abstract**

We propose a SUSY variant of the action for a massless spinning particles via the inclusion of twistor variables. The action is constructed to be invariant under SUSY transformations and $\tau$-reparametrizations even when an interaction field is including. The constraint analysis is achieved and the equations of motion are derived. The commutation relations obtained for the commuting spinor variables $\lambda_\alpha$ show that the particle states have fractional statistics and spin. At once we introduce a possible massive term for the non-interacting model.

## 1 Introduction

The field theory in space-time $D = 2 + 1$ has some interesting features related with the nontrivial topology of the configuration space. For example, solitons of $D = 2 + 1$ theories can hold fractional charge, statistics and spin [1, 2, 3, 4], many of such systems have been observed in condensed matter experiments.

Alternatively, other phenomenological models implement the appearance of exotic statistics by the addition of a Chern-Simons term to the effective action for a statistical gauge field [5, 6]. For example, such interesting situation ocurred in the $O(3)$ $\sigma-$ model proposed by Balachandran *et al.* [7], where the Chern-Simon term is constructed from the $SU(2)$ connection form on $\sigma-$ model fiber bundle space with $S^2$ sphere as the base, where the quantization of this model leads to obtain solitons with fractional spin. In the Semenoff´s work [8, 9] solitons with exotic statistical properties are also obtained when the interaction of the scalar and abelian gauge field is considered. Other works involving particles with fractional spin can be found in [10], where a non-Grassmannian approach is formulated on the pseudoclassical basis for the massive as well as for the massless case.

The problem to the construction of a consistent field theory for quartions in dimensions $D = 2+1$ and $D = 3+1$ was considered by Volkov *et al.* in the works [11, 12]. The extension of the free theory to higher dimensional space-times must be performed with a special care because there is a theorem which states that in $D \geq 3+1$ the statistics must be either fermionic or bosonic ones. As we know this theorem is valid for the finite-dimensional representation of the Lorentz group. However in the works [11, 12] it is showed that the fractional spin-states are described by the infinite-dimensional representations of the Lorentz group and the existence of quartions in higher dimensions is also possible. It is worthwhile to remark that in $D = 3 + 1$ a pair of linear independent equations is obtained and it becomes inconsistent when the interaction is included. However as Volkov *et al.* [11, 12] pointed out there is the possibility to describe the dynamic of quartions by means of the twistor variables and the interactions can be studied in a consistent way.

For further development of the theory, it would be very useful to establish the fundamental connection between space-time and twistor description of particles and superparticles at the Lagrangian level. Twistor theory has been developed mainly by Penrose [13, 14] and the theory is in fact largely based on ideas of conformal symmetry, i.e., zero rest mass particles and conformally invariant fields. In this formalism, the basic variables to describe the dynamics of the massless spinning particles are a pair of spinor variables called twistor and the procedure of canonical quantization can be applied to these variables. In this sense the space of twistors can be considered

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as more basic and fundamental than space-time and in certain cases it allows a simplification of the constraint analysis and a larger transparency of the symmetry properties. Consequently, when the twistor techniques [15, 16] are implemented into the structure of supersymmetric theories, a new ingredient to study the different models is presented.

The main goal in this present work is to explore the consequences of the vacuum fluctuation of one of these models [17] just originated by the twistor variables. For this purpose we give a SUSY generalization of this action and study the constraint structure of the model for the free case as well as when an interaction is included.

The paper is organized as follows: in section **2** we give a brief review about theories that consider particles with fractional spin and statistics (quartions). We discuss the connections between fractional statistics and fractional spin and see that the possibility of the existence of quartions no contradict the fundamental Pauli principle. In section **3** we start with the action for a free massless spinning particles in $D = 2 + 1$ that includes twistor variables. Next, considering only the vacuum fluctuations we construct an action that is invariant under SUSY transformations and $\tau$ - reparametrizations, in following we perform the constraint analysis for the free case as well as for an interacting "gauge" field and, finally a massive term to the model is introduced. In section **4** we give our final remarks and conclusions.

## 2 Particles with fractional spins

It was shown [11] that quantum field theories in $D = 2 + 1$ dimensions have a very interesting structure when the connection between statistical and spin properties is studied. As it was pointed out, the existence of objects (quartions) possessing nontrivial (exotic) spin no contradict the fundamental Pauli principle that establishes the existence of integer or half integer spin. The existence of quartions is concerning with the topological properties of the space-time and it is in complete agreement with the group-theoretical description of its dynamical properties.

The Poincare group (or the inhomogeneous Lorentz group $ISO(1,2)$) is constructed by three translation generators $P_m$ $(m = 0, 1, 2)$ and three angular momenta generators $M_m$ of the Lorentz group $SO(1,2)$ that is isomorphic to $SL(2, \mathbf{R})$. It is well known that the $ISO(1,2)$ generators satisfy the following commutation relations

$$[P_m, P_n] = 0, \quad [M^m, M^n] = i\epsilon^{mnl} M_l, \quad [M^m, P^n] = i\epsilon^{mnl} P_l \tag{1}$$

here $\epsilon^{mnl}$ is the total antisymmetric tensor and the space-time metric is defined by $\eta^{mn} = diag\,(+,-,-)$. There are three independent Casimir operators

$$\begin{aligned} C_1 &= P^n P_n = m^2 \\ C_2 &= M_n P^n \\ C_3 &= \frac{P_0}{|P_0|} \end{aligned} \tag{2}$$

where we see that the mass shell condition and the Pauli-Liubanski scalar are defined by the the two first relations while the third one is the energy sign.

A consistent relativistic field theory for particles with fractional spin and statistics is constructed on the base of the Heisenberg-Weyl group [18, 19] whose irreducible representations are given by the particle states with spin values $S_{1/4}$ and $S_{3/4}$. As it is known this group is generated by the coordinate $q$ and momentum $p = i\hbar\partial/\partial q$ operators acting on vectors of the Hilbert space satisfying the usual commutation relations

$$[q, p] = i, \quad [q, q] = [p, p] = 0 \tag{3}$$

We recall that in the considered theory $q$ parametrizes the quartion spin space. As customary the action of the rising $a^+$ and lowering $a$ operator

$$a^+ = \frac{1}{\sqrt{2}}\,(q - ip)\,, \quad a = \frac{1}{\sqrt{2}}\,(q + ip) \tag{4}$$

onto the vacuum vector $|0\rangle$ , generates the corresponding orthonormal basic vector of the representation space that has the following form

$$|n\rangle = (n!)^{-1/2} \left(a^+\right)^n |0\rangle\,, \quad n = 0, 1, 2, ... \tag{5}$$

Defining the Majorana spinor

$$L_\alpha = \left( \begin{array}{c} q \\ p \end{array} \right) \tag{6}$$

it is possible to construct the $SL(2,R)$ group generators by means of the Heisemberg-Weyl generators $q$, $p$ in a Lorentz covariant manner.

With this definition the commutation relation (3) becomes

$$[L_\alpha, L_\beta] = -i\hbar\epsilon_{\alpha\beta} \tag{7}$$

where $\epsilon_{\alpha\beta}$ is the antisymmetric matrix $\epsilon_{12} = 1$. The last relation determines, in our case, the nature of the theory under consideration and implies in the possible existence of particles with exotic spin and statistics (quartions).

The $SL(2,R)$ generators acting on the representations $S_{1/4}, S_{3/4}$ are given by the anticommutators of spinors $L_\alpha$ components as follows

$$M_{\alpha\beta} = iM_n (\gamma^n)_{\alpha\beta} = \frac{1}{4}(L_\alpha L_\beta + L_\beta L_\alpha) = \frac{1}{2}\{L_\alpha, L_\beta\} \tag{8}$$

As it is well known, spinors have a richer structure than vectors, and is connected with the group properties of the $SU(2)$ which is the covering group of the rotation group $O(3)$. In this sense the existence of quartions can be considered as more fundamental than spinors and should have a certain relation with the elementary particle physics.

As it was given in [11, 12] the equation for quartions in Lorentz covariant form can be written as

$$(L^\alpha P_{\alpha\beta} - mL_\beta)\Phi = 0 \tag{9}$$

and it resembles the Dirac equation if we put $L_\alpha \Phi = \Psi$, thus in our case $\Phi$ has a continuous dependence on the spin parameter. It is important to remark that in these models there are problems concerned with the construction of the Lagrangian which generates the equations of motions (9). Another problem, related to the development of the theory based on the equation (9) is the difficulty to add interactions of quartions with the common fields as, for example, the electromagnetic interaction that can be implemented via the minimal coupling procedure. Therefore, other alternatives must be explored to obtain a satisfactory and consistent theory for quartions. We will try to reach our goal by means of SUSY resources.

# 3 Relativistic Particle Dynamics

## 3.1 Free case

We begin with the formulation of massless relativistic particle dynamics in $D = 2+1$ - dimensional space-time. The momentum vector $p_{\alpha\beta} = \gamma^m_{\alpha\beta} p_m$ is written as a bilinear combination of twistor components $\lambda_\alpha$ obtaining for the proposed action [20]

$$S = \int d\tau \lambda_\alpha \lambda_\beta \dot{x}^{\alpha\beta} \tag{10}$$

which is the connection between the space-time formulation and the twistor one. Here $\lambda_\alpha$ is a commuting Majorana spinor, the index $\alpha, \beta = 1, 2$, $x^{\alpha\beta}(\tau) = \gamma^{m\alpha\beta} x_m(\tau)$ is the coordinate of the particle $(m = 0, 1, 2)$ and $\dot{x}^{\alpha\beta} = \frac{d}{d\tau} x^{\alpha\beta}(\tau)$.

The inclusion of twistor variables enables us to consider the vacuum fluctuations giving an additional term containing a $\dot{\lambda}^\alpha$ and that is considered minimally into the action [17]

$$S_0 = l \int d\tau \lambda_\alpha \dot{\lambda}^\alpha \tag{11}$$

where $l$ is an arbitrary parameter of length and was introduced to assure the correctness of the action dimension.

We consider the motion of the particle in the large superspace $(X_m, \Theta_\alpha)$ whose trajectory is parameterized by the proper supertime $(\tau, \eta)$ of dimension $(1/1)$ ($\eta$ is the grassmannian real superpartner of the conventional time $\tau$). In this way the coordinates of the particle trajectory constitute scalar superfields in the little superspace $(1/1)$:

$$X_m(\tau,\eta) = x_m(\tau) + i\eta\psi_m(\tau) \tag{12}$$
$$\Theta_\alpha(\tau,\eta) = \theta_\alpha(\tau) + \eta\lambda_\alpha(\tau) \tag{13}$$

where the grassmannian variable $\psi_m$ is the superpartner of the bosonic coordinate $x_m$ and the commuting Majorana spinor $\lambda_\alpha$ is the superpartner of the grassmannian variable $\theta_\alpha$.

In order to construct an action which is invariant under general transformations in superspace we introduce the supereinbein $E_M^A(\tau,\eta)$, where $M$ $[A]$ are curved [tangent] indices and $D_A = E_A^M \partial_M$ is the supercovariant general derivative, $E_A^M$ is the inverse of $E_M^A$. In the special gauge [23]

$$E_M^\alpha = \Lambda \overline{E}_M^\alpha, \quad E_M^a = \Lambda^{1/2}\overline{E}_M^a \tag{14}$$

where

$$\overline{E}_\mu^\alpha = 1, \quad \overline{E}_\mu^a = 0, \quad \overline{E}_m^\alpha = -i\eta, \quad \overline{E}_m^a = 1 \tag{15}$$

is the flat space supereinbein. In this case, the superscalar field $\Lambda$ and the derivative $D_A$ can be written as

$$\Lambda = e + i\eta\chi, \quad \overline{D}_a = \partial_\eta + i\eta\partial_\tau, \quad \overline{D}_\alpha = \partial_\tau \tag{16}$$

where $e(\tau)$ is the graviton field and $\chi(\tau)$ is the gravitino field of the 1 - dimensional $n = 1$ supergravity. There is no difficult to prove that

$$\left(\overline{D}_a\right)^2 \equiv (D_\eta)^2 = i\partial_\tau$$

The extension to superspace of the actions (10) and (11) is given by[1]

$$S = il\int d\tau d\eta \Lambda^{-1} D_\eta X^{\alpha\beta} D_\eta \Theta_\alpha D_\eta \Theta_\beta \tag{17}$$

and

$$S_0 = \frac{il}{2}\int d\tau d\eta \Lambda^{-1} D_\eta \Theta_\alpha \dot{\Theta}^\alpha, \tag{18}$$

respectively. Where we introduce the length constant $l$ to obtain the correct dimension of the superfield components, however, the final results will be $l$-independent.

From the condition $\Lambda\Lambda^{-1} = 1$ we obtain

$$\Lambda^{-1} = e^{-1} - ie^{-2}\eta\chi. \tag{19}$$

Our main goal is to study the dynamics of the action (18) arising when we consider the vacuum fluctuations. We also remark that $S_0$ appears due to the twistor variables introduced in the action (10).

After simple manipulations we obtain for the action (18) in the second order formalism

$$S_0 = l\int d\tau \left[\frac{1}{2}e^{-1}\left(i\dot{\theta}_\alpha\dot{\theta}^\alpha + \lambda_\alpha\dot{\lambda}^\alpha\right) - \frac{i}{2}e^{-2}\chi\lambda_\alpha\dot{\theta}^\alpha\right]. \tag{20}$$

Immediately, we do the following redefinition of the fields

$$\lambda_\alpha = e^{1/2}\widehat{\lambda}_\alpha, \quad \chi = e^{1/2}\widehat{\chi} \tag{21}$$

that allows to rewrite the action $S_0$ as being

$$S_0 = l\int_{\tau_1}^{\tau_2} d\tau \left[\frac{i}{2}e^{-1}\left(\dot{\theta}_\alpha - \frac{1}{2}\widehat{\chi}\widehat{\lambda}_\alpha\right)\left(\dot{\theta}^\alpha - \frac{1}{2}\widehat{\chi}\widehat{\lambda}^\alpha\right) + \frac{1}{2}\widehat{\lambda}_\alpha\dot{\widehat{\lambda}}^\alpha\right] + \frac{l}{2}\widehat{\lambda}_\alpha(\tau_2)\widehat{\lambda}^\alpha(\tau_1) \tag{22}$$

[1] As we will see later the presence of the superscalar field $\Lambda$ guarantees the local SUSY invariance.

note that the "small" supersymmetrization of the action (11) generates the kinetic term for the dynamical variable $\theta_\alpha$. The boundary term in (22) was introduced to get a set of consistent equations of motion which are given by

$$\dot{\widehat{\lambda}}_\alpha = \frac{ie^{-1}}{2}\widehat{\chi}\left(\dot{\theta}_\alpha - \frac{1}{2}\widehat{\chi}\widehat{\lambda}_\alpha\right), \quad \widehat{\lambda}^\alpha \pi_\alpha = 0, \quad \pi_\alpha \pi^\alpha = 0, \quad \dot{\pi}_\alpha = 0. \tag{23}$$

We follow the standard Dirac procedure to study the constrained system generated by the action **(22).** The canonical momentum obtained from (22) are

$$\pi_\alpha = ie^{-1}\left(\dot{\theta}_\alpha - \frac{1}{2}\widehat{\chi}\widehat{\lambda}_\alpha\right) \tag{24}$$

$$\varkappa_\alpha = \frac{1}{2}\widehat{\lambda}_\alpha, \quad \pi_\chi = 0, \quad \pi_e = 0. \tag{25}$$

The set of primary constraints is

$$\Omega_\alpha = \varkappa_\alpha - \frac{1}{2}\widehat{\lambda}_\alpha \approx 0, \quad \Omega_\chi = \pi_\chi \approx 0, \quad \Omega_e = \pi_e \approx 0 \tag{26}$$

The primary hamiltonian associated to the action (22) and that considers the primary constraints is given by

$$\mathcal{H}_P = -\frac{i}{2}e\pi_\alpha\pi^\alpha - \frac{1}{2}\widehat{\chi}\widehat{\lambda}^\alpha\pi_\alpha + \Gamma^a\Omega_a \tag{27}$$

where $\Gamma^a \equiv \{\Gamma^\alpha, \Gamma^\chi, \Gamma^e\}$ are the lagrange multipliers. The stability condition applied on the primary constraints gives a set of secondary constraints

$$\Omega_\chi^{(2)} = \frac{1}{2}\widehat{\lambda}^\alpha\pi_\alpha \approx 0, \quad \Omega_e^{(2)} = \frac{i}{2}\pi_\alpha\pi^\alpha \approx 0 \tag{28}$$

which yield a set of first class constraints. With the help of the second class constraint $\Omega_\alpha = \varkappa_\alpha - \frac{1}{2}\widehat{\lambda}_\alpha \approx 0$ we can construct the Dirac Bracket (DB) for any two variables

$$\{F, G\}_{DB} = \{F, G\}_{PB} - \{F, \Omega_\alpha\}_{PB} C^{-1}_{\alpha\beta} \{\Omega_\beta, G\}_{PB} \tag{29}$$

where $C_{\alpha\beta}$ is the matrix formed by the Poisson Bracket (PB) of the second class constraints. Thus we derive the DB for the canonical variables

$$\left\{\theta^\alpha, \theta^\beta\right\}_{DB} = \{\pi_\alpha, \pi_\beta\}_{DB} = 0 \tag{30}$$

$$\{\theta^\alpha, \pi_\beta\}_{DB} = -\delta_{\alpha\beta}, \quad \left\{\widehat{\lambda}_\alpha, \widehat{\lambda}_\beta\right\}_{DB} = \epsilon_{\alpha\beta} \tag{31}$$

There are two types of gauge (super) transformations that leave the action (22) invariant: The invariance under local SUSY transformations

$$\begin{aligned} \delta\theta_\alpha &= \alpha(\tau)\widehat{\lambda}_\alpha, \quad \delta\widehat{\lambda}_\alpha = i\alpha(\tau)e^{-1}\left(\dot{\theta}_\alpha - \frac{1}{2}\widehat{\chi}\widehat{\lambda}_\alpha\right) \\ \delta e &= i\alpha(\tau)\widehat{\chi}, \quad \delta\widehat{\chi} = 2\dot{\alpha}(\tau) \end{aligned} \tag{32}$$

and the $\tau$-reparametrizations

$$\begin{aligned} \delta\theta_\alpha &= a(\tau)\dot{\theta}_\alpha, \quad \delta\widehat{\lambda}_\alpha = a(\tau)\dot{\widehat{\lambda}}_\alpha \\ \delta e &= (ae)^{\cdot}, \quad \delta\widehat{\chi} = (a\widehat{\chi})^{\cdot} \end{aligned} \tag{33}$$

The invariance under $\tau$-reparametrizations is required by the fact that we can choose any parameter without altering the physics of the system.

It is interesting to commute two SUSY transformations, then, using (32) we obtain

$$\begin{aligned} [\delta_\alpha, \delta_\beta]\, \theta_\alpha &= f\dot{\theta}_\alpha + \overline{\delta}_g \theta_\alpha, \quad [\delta_\alpha, \delta_\beta]\, \widehat{\lambda}_\alpha = f\dot{\widehat{\lambda}}_\alpha + \overline{\delta}_g \widehat{\lambda}_\alpha \\ [\delta_\alpha, \delta_\beta]\, e &= (fe)^{\cdot} + \overline{\delta}_g e, \quad [\delta_\alpha, \delta_\beta]\, \widehat{\chi} = (f\widehat{\chi})^{\cdot} + \overline{\delta}_g \widehat{\chi} \end{aligned} \tag{34}$$

where we have introduced a new reparametrization ($f$) and SUSY ($g$) transformation parameters

$$f(\tau) = 2i\beta\alpha e^{-1}, \quad g(\tau) = -\frac{1}{2} f\widehat{\chi} \tag{35}$$

Thus we see that the commutation of two SUSY transformations yields a reparametrization (with parameter $f$) plus an additional SUSY transformation (with parameter $g$). We also remark that the new transformation parameters are field dependent.

The generator $G$ of the transformations (32) and (33) can be found by means of [21, 22]

$$\epsilon G = p_a \delta a^a - \varphi, \quad \delta L = \frac{d\varphi}{d\tau} \tag{36}$$

where $\epsilon^a$ are the transformation parameters and, $\varphi$ is the generating function. The generators must satisfy the relation

$$\delta u = \{u, \epsilon G\}_{DB} \tag{37}$$

being $u$ any of the coordinate $q^a$.

In this way, we get for the the local SUSY transformations

$$\begin{aligned} G &= -\widehat{\lambda}^\alpha \widehat{\pi}_\alpha + i\widehat{\chi}\pi_e \\ \{\theta^\alpha, \alpha G\}_{DB} &= \alpha\widehat{\lambda}^\alpha, \quad \left\{\widehat{\lambda}^\alpha, \alpha G\right\}_{DB} = i\alpha\left(\dot{\theta}^\alpha - \frac{1}{2}\widehat{\chi}\widehat{\lambda}^\alpha\right) \\ \{e, \alpha G\}_{DB} &= i\alpha\widehat{\chi} \end{aligned} \tag{38}$$

and the following $\tau$-reparametrizations

$$\begin{aligned} G &= -\frac{1}{2} e\pi_\alpha \pi^\alpha - \frac{1}{2}\widehat{\chi}\widehat{\lambda}^\alpha \pi_\alpha \\ \{\theta^\alpha, aG\}_{DB} &= \alpha\dot{\theta}^\alpha, \quad \left\{\widehat{\lambda}^\alpha, aG\right\}_{DB} = a\dot{\widehat{\lambda}}^\alpha \end{aligned} \tag{39}$$

the last result shows that the canonical hamiltonian is the generator of the $\tau$-reparametrizations.

## 3.2 Quantization

The quantization of the model is performed using the correspondence principle where the Dirac brackets of the dynamical variables transform in commutator or anticommutator $\{\widehat{\ \ }\} \to \frac{\hbar}{i}\{\quad\}_{DB}$ , i.e.

$$\left\{\widehat{\theta}^\alpha, \widehat{\theta}^\beta\right\} = \{\widehat{\pi}_\alpha, \widehat{\pi}_\beta\} = 0 \tag{40}$$

$$\left\{\widehat{\theta}^\alpha, \widehat{\pi}_\beta\right\} = ih\delta_{\alpha\beta}, \quad \left[\widehat{\lambda}_\alpha, \widehat{\lambda}_\beta\right] = -ih\epsilon_{\alpha\beta}. \tag{41}$$

The first class constraints are applied on the quartion vector states $|\Phi\rangle$

$$\widehat{\lambda}_\alpha \widehat{\pi}^\alpha |\Phi\rangle = 0 \tag{42}$$

$$\widehat{\pi}_\alpha \widehat{\pi}^\alpha |\Phi\rangle = 0. \tag{43}$$

After a simple manipulation we can see that $\left(\widehat{\lambda}_\alpha \widehat{\pi}^\alpha\right)^2 \approx \widehat{\pi}_\alpha \widehat{\pi}^\alpha$. In a certain sense this leads to interpret the (42) as the Dirac equation and the (43) as the Klein-Gordon equation. However it is necessary to point out that in this model we do not have necessarily particles with spin 1/2 or 0.

Immediately, we select a particular realization for the operators satisfying the commutation relations (40) and (41)

$$\mathcal{D}\left(\widehat{\theta}_\alpha\right) = \theta_\alpha, \quad ,\mathcal{D}\left(\widehat{\lambda}_\alpha\right) = L_\alpha \tag{44}$$

$$\mathcal{D}\left(\widehat{\pi}_\alpha\right) = i\hbar\frac{\partial}{\partial\theta^\alpha} \equiv i\hbar\partial_\alpha \tag{45}$$

where $L_\alpha$ is the operator given in (6) just the realization for the operators that describes particles with exotic spin (quartions). This result enables us to consider the presence of quartions inside the vector state $|\Phi\rangle$ and, a possible supermultiplet formed by particles with spin $s = 1/4, 3/4$. We emphasize that it does not contradict the SUSY principles since the difference between the minimal weight is equal to $1/2$ just as it happened in any SUSY transformation.

### 3.3 Interaction

Now we will analyze our system when a "gauge" field is added. To construct the action that includes the interaction of the vacuum fluctuations with a certain gauge field must be considered their functional nature. Then the action takes the form [24]

$$S_1 = ig\int d\tau d\eta D_\eta \Theta_\alpha \mathbf{A}^\alpha\left(\Theta\right) \tag{46}$$

where $g$ is the coupling constant for interaction and $\mathbf{A}^\alpha\left(\Theta\right)$ is a "functional" supergauge field given by

$$\mathbf{A}^\alpha\left(\Theta\right) \equiv \mathbf{A}^\alpha\left(\theta,\eta;\lambda\right) = A^\alpha\left(\theta\right) + \eta B^\alpha\left(\theta;\lambda\right) \tag{47}$$

with $A^\alpha$ being the grassmannian superpartner of the bosonic field $B^\alpha$. On the other hand, considering (13) we obtain

$$\mathbf{A}^\alpha\left(\Theta\right) \equiv \mathbf{A}^\alpha\left(\theta + \eta\lambda\right) = A^\alpha\left(\theta\right) + \eta\lambda_\beta\frac{F^{\beta\alpha}\left(\theta\right)}{2} \tag{48}$$

the factor $\frac{1}{2}$ in the last relation is inserted for convenience. From (47) and (48), we conclude that

$$B^\alpha\left(\theta;\lambda\right) = \frac{1}{2}\lambda_\beta F^{\beta\alpha}\left(\theta\right) \tag{49}$$

using the equations (13), (16) and (49) we can write the action (46) as being

$$S_1 = ig\int d\tau\left(e\widehat{\lambda}_\alpha\widehat{B}^\alpha + \dot{\theta}_\alpha A^\alpha\right) = ig\int d\tau\left(\frac{1}{2}e\widehat{\lambda}_\alpha\widehat{\lambda}_\beta F^{\beta\alpha} + \dot{\theta}_\alpha A^\alpha\right) \tag{50}$$

where we have redefined the fields as in (21). Due the commutation relation for the spinor $\widehat{\lambda}_\alpha$ we infer that only the symmetrical part of the field $F^{\beta\alpha}$ contributes to this action.

The action (50) is invariant under local SUSY transformations (32) with

$$\delta A^\alpha = i\alpha\left(\tau\right)\widehat{B}^\alpha = \frac{i}{2}\alpha\left(\tau\right)\widehat{\lambda}_\beta F^{\beta\alpha} \tag{51}$$

$$\delta\widehat{B}^\alpha = i\alpha\left(\tau\right)e^{-1}\left[\dot{A}^\alpha - \frac{i}{2}\widehat{\chi}\widehat{B}^\alpha\right] \tag{52}$$

this invariance provides an unique value for the field $F^{\alpha\beta}$ which results in

$$F^{\alpha\beta} = i\left(\partial^\beta A^\alpha + \partial^\alpha A^\beta\right) \tag{53}$$

it is no difficult to show that

$$\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0 \tag{54}$$

On account of the connection of the $SL(2,R)$ and $O(3)$ groups where the $\sigma^m$ matrices play the role of Clebsh-Gordon coefficients, we infer the following relation between the quantities $F_{\alpha\beta}$ and $F_{mn}$

$$\begin{aligned} F^{mn} &= (\sigma^{mn})_{\alpha\beta} F^{\alpha\beta}, \quad \partial_m A^m = \partial_{\alpha\beta} F^{\alpha\beta} = 0 \\ F^{mn} &= \partial^m A^n - \partial^n A^m \end{aligned} \tag{55}$$

i.e. $F^{\alpha\beta}$ is the spinor form of the "electromagnetic field". We remember that for this "spin tensor" field in $D = 2+1$ dimensions there are only 3 linearly independent components.

The invariance under $\tau$-reparametrizations (33) are completed with

$$\delta A^\alpha = a\dot{A}^\alpha \tag{56}$$

$$\delta \widehat{B}^\alpha = a\dot{\widehat{B}}^\alpha \tag{57}$$

Joining the free action (22) with the interaction action (50) we have

$$\begin{aligned} S &= \int d\tau \left[ \frac{i}{2} e^{-1} \left( \dot{\theta}_\alpha - \frac{1}{2} \widehat{\chi} \widehat{\lambda}_\alpha \right) \left( \dot{\theta}^\alpha - \frac{1}{2} \widehat{\chi} \widehat{\lambda}^\alpha \right) + \frac{1}{2} \widehat{\lambda}_\alpha \dot{\widehat{\lambda}}^\alpha \right. \\ &\quad \left. + \frac{i}{2} eg \widehat{\lambda}_\alpha \widehat{\lambda}_\beta F^{\beta\alpha} + ig \dot{\theta}_\alpha A^\alpha \right]. \end{aligned} \tag{58}$$

From which we obtain the following canonical momentum conjugate

$$\begin{aligned} \pi_\alpha &= ie^{-1} \left( \dot{\theta}_\alpha - \frac{1}{2} \widehat{\chi} \widehat{\lambda}_\alpha \right) + iA_\alpha = \mathcal{P}_\alpha + igA_\alpha \\ \varkappa_\alpha &= \frac{1}{2} \widehat{\lambda}_\alpha, \quad \pi_\chi = 0, \quad \pi_e = 0, \quad \pi_\alpha^A = 0, \quad \pi_\alpha^B = 0 \end{aligned} \tag{59}$$

and the primary constraints

$$\begin{aligned} \Omega_\alpha &= \varkappa_\alpha - \frac{1}{2} \widehat{\lambda}_\alpha \approx 0, \quad \Omega_\chi = \pi_\chi \approx 0, \quad \Omega_e = \pi_e \approx 0 \\ \Omega_\alpha^A &= \pi_\alpha^A \approx 0, \quad \Omega_\alpha^B = \pi_\alpha^B \approx 0 \end{aligned} \tag{60}$$

The extended hamiltonian that considers the primary constraints (60) is

$$\mathcal{H}_P = -\frac{i}{2} e \mathcal{P}_\alpha \mathcal{P}^\alpha - \frac{i}{2} eg \widehat{\lambda}_\alpha \widehat{\lambda}_\beta F^{\alpha\beta} + \frac{1}{2} \widehat{\chi} \widehat{\lambda}_\alpha \mathcal{P}^\alpha + \Gamma^a \Omega_a \tag{61}$$

where $\Gamma^a \equiv \{\Gamma^\alpha, \Gamma^\chi, \Gamma^e, \Gamma_A^\alpha, \Gamma_B^\alpha\}$ are the new lagrange multipliers. The conservation of primary constraints in time leads to

$$T_2 \equiv \frac{1}{2} \widehat{\lambda}_\alpha \mathcal{P}^\alpha \approx 0, \quad T_1 \equiv \frac{i}{2} \left( \mathcal{P}_\alpha \mathcal{P}^\alpha + g \widehat{\lambda}_\alpha \widehat{\lambda}_\beta F^{\alpha\beta} \right) \approx 0 \tag{62}$$

which is a set of first class constraints satisfying the algebra

$$\{T_1, T_2\}_{DB} = 0, \quad \{T_1, T_1\}_{DB} = 0, \quad \{T_2, T_2\}_{DB} = \frac{i}{2} T_1. \tag{63}$$

In the same manner as in (29) we define the DB, that results in

$$\begin{aligned} \left\{ \theta^\alpha, \theta^\beta \right\}_{DB} &= 0, \quad \{\theta^\alpha, \mathcal{P}_\beta\}_{DB} = -\delta_{\alpha\beta} \\ \{\mathcal{P}_\alpha, \mathcal{P}_\beta\}_{DB} &= -gF_{\alpha\beta}, \quad \left\{ \widehat{\lambda}_\alpha, \widehat{\lambda}_\beta \right\}_{DB} = \epsilon_{\alpha\beta} \end{aligned} \tag{64}$$

Upon quantization the canonical variables become operators and the DB follows the commutator or anticommutator rules

$$\begin{aligned} \left\{ \widehat{\theta}^\alpha, \widehat{\theta}^\beta \right\} &= 0, \quad \left\{ \widehat{\theta}^\alpha, \widehat{\mathcal{P}}_\beta \right\} = i\hbar \epsilon_{\alpha\beta} \\ \left\{ \widehat{\mathcal{P}}_\alpha, \widehat{\mathcal{P}}_\beta \right\} &= i\hbar g F_{\alpha\beta}, \quad \left[ \widehat{\lambda}_\alpha, \widehat{\lambda}_\beta \right] = -i\hbar \epsilon_{\alpha\beta}. \end{aligned} \tag{65}$$

The first class constraints are applied on the vector state $|\Phi\rangle$

$$\begin{aligned} \widehat{\lambda}_\alpha \mathcal{P}^\alpha |\Phi\rangle &= 0 &(66)\\ \left(\mathcal{P}_\alpha \mathcal{P}^\alpha + g\widehat{\lambda}_\alpha \widehat{\lambda}_\beta F^{\alpha\beta}\right)|\Phi\rangle &= 0 &(67)\end{aligned}$$

We note that the first equation (66) obeys the minimal coupling principle when a gauge field is added. On the other hand (67) is the Klein-Gordon-Fock equation when the interaction is considered.

The possible realization for the resulting operators that take into account the commutation relations (65) is similar to the free case (44) but the equation (45) suffers a compatible modification with the minimal coupling principle,

$$\mathcal{D}\left(\widehat{\mathcal{P}}_\alpha\right) = i\hbar\frac{\partial}{\partial\theta^\alpha} + iA_\alpha \equiv i\hbar\partial_\alpha + igA_\alpha. \qquad (68)$$

As the representations (44) remain the same, the possibility to obtain quartions in our analysis is maintained.

### 3.4 The massive Term

We consider the possibility of including a massive term to the lagrangian (20). The SUSY extension for this term is non trivial and requires concepts and methods of spontaneous SUSY breaking. Nevertheless, we give a possible component form of the model based on ideas of the pseudoclassical formalism, thus, a consistent action including a massive term is given by

$$S_m = \frac{i}{2}\int_{\tau_1}^{\tau_2} d\tau \left(em^2 + i\theta_5\dot{\theta}_5 + im\widehat{\chi}\theta_5\right) + \frac{i}{2}\theta_5(\tau_2)\theta_5(\tau_1) \qquad (69)$$

where $\theta_5$ is a grassmannian variable and the boundary term is added for the consistence of the resulting equation of motions. The action (69) preserves the invariance under local SUSY transformations (32) and $\tau$-reparametrizations (33) when $\delta\theta_5 = m\alpha$ and $\delta\theta_5 = a\dot{\theta}_5$ are included, respectively. Thus the new hamiltonian for the massive free case results in

$$\mathcal{H} = -\frac{ie}{2}\left(\pi_\alpha\pi^\alpha + m^2\right) - \frac{1}{2}\widehat{\chi}\left(\widehat{\lambda}^\alpha\pi_\alpha - m\lambda_5\right) \qquad (70)$$

The constraint analysis of the new system provides the following set of first class constraints

$$\pi_\alpha\pi^\alpha + m^2 \approx 0, \quad \widehat{\lambda}^\alpha\pi_\alpha - m\theta_5 \approx 0 \qquad (71)$$

and second class constraints

$$\varkappa_\alpha - \frac{1}{2}\widehat{\lambda}_\alpha \approx 0, \quad \varkappa_5 - \frac{1}{2}\theta_5 \approx 0. \qquad (72)$$

## 4 Conclusions

In this work we have constructed in $D = 2+1$ dimensional space-time a supersymmetric version of the action that describes the vacuum fluctuations of the massless relativistic particles, these contribution appears when twistor variables are introduced in the theory [17]. The construction is performed leaving the action invariant under local SUSY transformations and $\tau$- reparametrizations. The general Dirac procedure to the analysis of constrained systems was performed obtaining after quantization a very interesting result, i.e., the possibility to appear particles states with fractional spin. Our result is preserved even when a certain "gauge" superfield $A_\alpha$ is switched on. We argued that the proposed action via inclusion of twistor variables also give a consistent method to study interactions of quartions and "gauge" fields. The multiplet formed by this particles is in complete accordance with the SUSY principles because the difference between the minimal weights (spins) in the multiplet is equal to 1/2.

On the other hand we have included a massive term to the studied action (11). The SUSY extension for this term is non trivial and requires concepts and methods of spontaneous SUSY breaking. Nevertheless, we give a possible component form of the model based on ideas of pseudoclassical formalism by the introduction of the grassmannian variable $\theta_5$. The contribution must be added to the action preserving its invariance under local SUSY transformations and $\tau$-reparametrizations. The study of the meaning of a massive theory for quartions and exploration of the resulting multiplet will also be explored.

Further we will study the extension of the model to $D = 3 + 1$ dimension and will also explore the possibility of obtaining particles with fractional statistics and spin. Here we point out that this requires the use of the covering group $SL(2, C)$ and must be considered two types of spinors $(\alpha, \dot{\alpha})$. This implies that the contribution to the vacuum fluctuation will have the additional term $\lambda_{\dot{\alpha}} \dot{\lambda}^{\dot{\alpha}}$ and the existence of the antiparticles could arise in this model.

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