Convex Regularization of Local Volatility Models from Option Prices: Convergence Analysis and Rates

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Joint work with **J. P. Zubelli (IMPA)** and **O. Scherzer (Univ. Vienna)**

Plan

Derivative Contracts

European Call Option: a forward contract that gives the holder the right, but not the obligation, to buy one unit of an underlying asset for an agreed *strike price* K on the *maturity* date T.

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European Call Option: a forward contract that gives the holder the right, but not the obligation, to buy one unit of an underlying asset for an agreed *strike price* K on the *maturity* date T.
Its payoff is given by

$$h(X_T) = \begin{cases} X_T - K & \text{if } X_T > K, \\ 0 & \text{if } X_T \leq K. \end{cases}$$

Fundamental Question *How to price such an obligation given today's information?*

Main Contributions

- L. Bachelier (Paris)
- P. Samuelson
- F. Black
- M. Scholes
- R. Merton

recognized by the Nobel prize in Economics award to R. Merton and M. Scholes

Black-Scholes Market Model

We consider the Black-Scholes equation (see Black-Scholes-1973):

$$U_t + \frac{1}{2}\sigma^2(t,X)X^2 U_{XX} + (r-q)XU_X = rU\,, \qquad (1)$$

$$U^{t,S}(t=T,X) = \max(X-K,0) \qquad \textbf{European Call} \qquad (2)$$

Here, $X = X(t)$ denotes the spot price, K is called strike, T the maturity, r is the interesting rate, q the dividends and $\sigma(t,X)$ is the local volatility.

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Note 2 Final Value Problem

Remark Constant volatility model - Nobel Prize: Robert Merton and Myron Scholes

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Since 1987 it is a decreasing function in the range $95\% < K/X < 105\%$ then (for $K >> X$) it bends upwards.
Limitations ...

Local Volatility Models

Idea Assume that the volatility is given by

$$\sigma = \sigma(t, x)$$

i.e.: it depends on time and the asset price.

The Direct Problem

Given $\sigma = \sigma(t, x)$ and the payoff information, determine $U = U(t, x, T, K; \sigma)$

The Inverse Problem

Given a set of observed prices

$$\{U = U(t, x, T, K; \sigma)\}_{(T,K)\in\mathcal{S}}$$

find the volatility $\sigma = \sigma(t, x)$.
The set $\mathcal{S}$ is taken typically as $[T_1, T_2] \times [K_1, K_2]$.
In Practice Very limited and scarce data

The Smile Curve and Dupire's Equation

Assuming that there exists a local volatility function $\sigma = \sigma(S,t)$ Dupire(1994) showed that the call price satisfies

$$\begin{cases} \partial_T U - \frac{1}{2}\sigma^2(K,T)K^2\partial_K^2 U + rS\partial_K U = 0\,, \quad S > 0\,, t < T \\ U(K, T=0) = (S-K)^+\,, \end{cases} \tag{3}$$

Theoretical way of evaluating the local volatility

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Theoretical way of evaluating the local volatility

$$\sigma(K,T) = \sqrt{2\left(\frac{\partial_T U + rK\partial_K U}{K^2\partial_K^2 U}\right)} \tag{4}$$

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Early Results

Bouchoev-Isakov Uniqueness and "Stability"

Consider the case of time-independent volatility and working with $y = \log(K/S(0)) \quad \tau = T - t$. Suppose $U(y,\tau)$ and $a(y) = \sigma(K)$ satisfies (3) with

$$U(y,0) = S(0)(1-e^y)^+, y \in \mathbb{R} \tag{5}$$

$$U(y,\tau^*) = U^*(y), y \in I \tag{6}$$

where I is a sub-interval of $\mathbb{R}$. Then, we have

- Uniqueness of the volatility
- Stability of the volatility in the Hölder λ-norm w.r.t. to variations of the data on the $2+\lambda$-norm. (i.e., one needs TWO extra derivatives of the data).

Early Results

More precisely

Theorem (Bouchouev & Isakov)

Let U_1 and U_2 be solutions of (3-6) with $a = a_1$ and $a = a_2$, resp., and the corresponding final data in Eq. (6) given by U_1^* and U_2^*. Let I_0 be an open interval with $I \supset I_0 \neq \emptyset$. Then,

1. *If $U_1^* = U_2^*$ on I and $a_1(y) = a_2(y)$ on I_0 then $a_1(y) = a_2(y)$ on I.*
2. *If, in addition, $a_1(y) = a_2(y)$ on $I_0 \cup (\mathbb{R} \setminus I)$ and I is bounded, then $\exists C = C(|a_1|_\lambda(I), |a_2|_\lambda(I), I, I_0, \tau^*)$ s.t.*

$$|a_1 - a_2|_\lambda(I) \leq |U_1^* - U_2^*|_{2+\lambda}(I)$$

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$$|a_1 - a_2|_\lambda(I) \leq |U_1^* - U_2^*|_{2+\lambda}(I)$$

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Start from

$$-U_t + \frac{1}{2}\sigma^2(T,K)K^2 U_{KK} - (r-q)KU_K - qU = 0 \qquad (7)$$

$$U^{t,X}(T=t,K) = (X-K)^+, \quad \text{for} \quad K>0, \qquad (8)$$

Setting $K = Xe^y$, $\tau = T-t$, $u(\tau,y) = \exp(\int_t^T q(s)ds)U(.,T,K)$,

$b(\tau) = q(\tau) - r(\tau)$ and

$$a(\tau,y) = \frac{1}{2}\sigma^2(\tau;K) \qquad (9)$$

yields

$$-u_\tau + a(\tau, y)(u_{yy} - u_y) + b(\tau)u_y = 0 \tag{10}$$

$$u(0, y) = X(1 - e^y)^+. \tag{11}$$

The parameter-to-solution map F is defined by

$$F(a) = u(a), \tag{12}$$

where $u(a)$ is the solution of (10) for $a \in \mathcal{D}(F)$.
We assume noise data u^δ satisfies the inequality

$$||u^* - u^\delta||_{L^2(\Omega)} \leq \delta. \tag{13}$$

Properties of F and ill-posedness of the inverse problem

i) There exists a $p^* > 2$ such that $F : \mathcal{D}(F) \to W_p^{2,1}(\Omega)$ is continuous and compact for $2 \leq p < p^*$. Moreover, F is weakly (sequentially) continuous and thus weakly closed.

ii) F is Gateaux differentiable w.r.t. $a \in \mathcal{D}(F)$ in directions h such that $a + h \in \mathcal{D}(F)$, and the derivative $F'(a)$ extends as a linear operator to $H^1(\Omega)$.

iii) $F'(a)$ is injective and compact. Moreover $\overline{\mathcal{R}(F'(a)^*)} = H^1(\Omega)$.

Proof: see DC & Scherzer & Zubelli (2009) and Egger & Engl (2005) or Crepey (2003).

Ill-posedness in more details

- $\mathcal{D}(F) := \{a \in a_0 + H^1 \,:\, \underline{a} \leq a, a_0 \leq \overline{a}\}$.
- $a_k \rightharpoonup \tilde{a}$ with $u_k = F(a_k) \to \tilde{u} = F(\tilde{a})$
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- ask for regularization!!!!
- Tikhonov regularization

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- source wise condition equivalent what we assume know about the inverse solution
- convergence rates to Tikhonov for quadratic penalization $||a^\delta - a^\dagger|| = O(\sqrt{\delta})$
- Our approach: convex regularization $\beta f(\cdot)$ with f convex, positive, etc...

State of the art from the literature

Let the standard choice of regularization parameter $\beta = \beta(\delta) \sim \delta$.

(i) If $f(\cdot) = ||\cdot||^2_{H^1(\Omega)}$, then we have the convergence rate results (Egger & Engl(2005))

$$||a^\delta_\beta - a^\dagger||_{H^1(\Omega)} = O(\sqrt{\delta}) \text{ and } ||F(a^\delta_\beta) - u^\delta||_{L^2(\Omega)} = O(\delta), \quad (14)$$

assuming the source-wise representation $a^* - a^\dagger = F'(a^\dagger)^* w$.

(ii) In (Hoffman & Kramer (2005)), with $f(a(\tau)) = \int_I \{a(\tau) ln\frac{a(\tau)}{\bar{a}(\tau)} + \bar{a}(\tau) - a(\tau)\} d\tau$ the convergence rate results

$$||a^\delta_\beta - a^\dagger||_{L^1([0,T])} = O(\sqrt{\delta}) \quad (15)$$

using a source-wise representation $ln\frac{a^\dagger}{a^*} = F'(a^\dagger)^* w$.

iii) With $f(\cdot) = ||\cdot||^2_{L^2([0,T])}$, (Hoffman et al. (2006)) obtained the same rates of (Hoffman & Kramer (2005) (volatility time-dependent only) assuming the source wise representation

$$\xi^\dagger = F'(a^\dagger)^* w \in \partial f(a^\dagger) \tag{16}$$

in terms of Bregman distances.

Our approach

Minimize the functional

$$\mathcal{F}_{\beta,u^\delta}(a) := \frac{1}{2}||F(a) - u^\delta||^2_{L^2(\Omega)} + \beta f(a), \tag{17}$$

where $f : dom(f) \subset \mathbb{B}_1 \longrightarrow [0,\infty]$ is a convex, proper and lower semi-continuous stabilization functional.

Convergence

Theorem (Existence, Stability, Convergence)

Suppose that F, f, $\mathcal{D}(F)$ as above, $\beta > 0$ and (13) holds. Then

- *There exists a minimizer of $\mathcal{F}_{\beta,u^\delta}$.*
- *If $(u_k) \to u$ in $L^2(\Omega)$, then every sequence (a_k) with*

$$a_k \in argmin\left\{\mathcal{F}_{\beta,u_k}(a) : a \in \mathcal{D}(F)\right\}$$

has a subsequence which weak converges. The limit of every w-convergent subsequence $(a_{k'})$ of (a_k) is a minimizer $\tilde{a}$ of $\mathcal{F}_{\beta,u}$, and $\left(f(a_{k'})\right)$ converges to $f(\tilde{a})$.

Theorem (Semi-convergence)

- *If there exists a solution of* (12) *in* $\mathcal{D}(F)$, *then there exists an* f*-minimizing solution of* (12).
- *Assume that* (12) *has a solution in* $\mathcal{D}(F)$ *(which implies the existence of an* f*-minimizing solution) and that* $\beta : (0,\infty) \to (0,\infty)$ *satisfies*

$$\beta(\delta) \to 0 \text{ and } \frac{\delta^2}{\beta(\delta)} \to 0\,, \text{ as } \delta \to 0\,. \tag{18}$$

Moreover, $(\delta_k) \to 0$, *and that* $u_k := u^{\delta_k}$ *satisfies* $\|\bar{u} - u_k\| \leq \delta_k$. *Set* $\beta_k := \beta(\delta_k)$. *Then, every sequence* (a_k) *of elements minimizing* $\mathcal{F}_{\beta_k, u_k}$, *has a subsequence* $(a_{k'})$ *that is w-convergent. The limit* $a^\dagger$ *of any w-convergent subsequence* $(a_{k'})$ *is an* f*-minimizing solution of* (12), *and* $f(a_k) \to f(a^\dagger)$.

Lemma

Let $\zeta^\dagger \in \partial f(a^\dagger)$. Then There exists a function $w^\dagger \in L^2(\Omega)$ and a function $r \in H^1(\Omega)$ such that

$$\zeta^\dagger = F'(a^\dagger)^* w^\dagger + r \tag{19}$$

holds. Furthermore, $\|r\|_{H^1(\Omega)}$ can to be taken arbitrarily small.

Definition

Let $1 \leq q < \infty$. Moreover, let $\tilde{U}$ be a subset of H^1. The Bregman distance $D_\zeta(\cdot, \tilde{a})$ of $f : H^1 \to \mathbb{R} \cup \{+\infty\}$ at $\tilde{a} \in \mathcal{D}_B(f)$ and $\zeta \in \partial f$ is said to be q-coercive with constant $\underline{c} > 0$ if

$$D_\zeta(a, \tilde{a}) \geq \underline{c} \|a - \tilde{a}\|_{\tilde{U}}^q \quad \forall a \in \mathcal{D}(f). \tag{20}$$

Convergence rates

Lemma

Let $\zeta^\dagger \in \partial f(a^\dagger)$ satisfy (19) with $w^\dagger$ and r such that

$$\underline{c}\left(C\|w^\dagger\|_{L^1(\Omega)} + \|r\|_{L^2(\Omega)}\right) := \beta_1 \in [0,1),$$

and the Bregman distance with respect to f is $1-$coercive (as in the Definition 6.1) with $\tilde{U} := H^1(\Omega)$. Then,

$$\langle \zeta^\dagger, a^\dagger - a\rangle \leq \beta_1 D_{\zeta^\dagger}(a, a^\dagger) + \beta_2 \|F(a) - F(a^\dagger)\|_{L^2(\Omega)}, \qquad (21)$$

for $a \in \mathcal{M}_{\beta_{max}}(\rho)$, where β_{max}, $\rho > 0$ satisfy the relation $\rho > \beta_{max} f(a^\dagger)$.

Theorem (Convergence rates)

Let $\beta : (0,\infty) \to (0,\infty)$ *satisfy* $\beta(\delta) \sim \delta$ *and* (21) *satisfied. Then,*

$$D_{\zeta^\dagger}(a_\beta^\delta, a^\dagger) = O(\delta)\,, \quad \text{and} \quad \|F(a_\beta^\delta) - u^\delta\|_{L^2(\Omega)} = O(\delta)\,,$$

and there exists $c > 0$, *such that* $f(a_\beta^\delta) \leq f(a^\dagger) + \delta / c$ *for every* δ *with* $\beta(\delta) \leq \beta_{max}$.

Proof: See DC & Scherzer & Zubelli (2009).

Example (q-coercive Bregman distance)

i) $f(a) := q^{-1}\|a - a^{\dagger}\|^{q}_{\tilde{U}}$.

ii) *Let* $1 < q \leq 2$. *We consider the functional*

$$f(a) = \sum_{n=1}^{\infty} | < a, \phi_n > |^q ,$$

where $\{\phi_n\}$ *is an orthonormal basis in* $H^1(\Omega)$. *The Bregman distance of the functional* f *satisfies*

$$f(a) - f(a^{\dagger}) - \langle \partial f(a^{\dagger}), a - a^{\dagger} \rangle \geq C \sum_{n=1}^{\infty} |\langle a - a^{\dagger}, \phi_n \rangle|^2 = C\|a - a^{\dagger}\|^2_{H^1(\Omega)} .$$

Exponential Families

Definition (Regular Exponential Family)

Let $\psi : \mathbb{R} \to \mathbb{R}_+$ *be convex and* $p_0 : \mathbb{R} \to \mathbb{R}_+$ *by continuous. The family of functions* $p_{\psi,\theta} : \mathbb{R} \to \mathbb{R}_+$ *defined by*

$$p_{\psi,\theta}(s) := \exp(s \cdot \theta - \psi(\theta)) p_0(s) \qquad \text{(regular exponential family).}$$

Theorem (Banerjee et al. [?])

Let ψ^* the Fenchel transform of ψ (differentiable) and $a(\theta) \in int(dom(\psi^*))$. Then,

$$p_{\psi,\theta}(a) = \exp\left(-D_{\psi^*}(a, a(\theta))\right)\exp\left(\psi^*(a)\right)p_0(a)\,. \tag{22}$$

Exponential Families and Fenchel conjugate

Example (Exponential Families and Fenchel conjugate)

Gaussian distribution $\psi(\theta) = \frac{\varpi^2}{2}\theta^2$*, then* $\psi^*(a) = \frac{a^2}{2\varpi^2}$*. Poisson distribution* $\psi(\theta) = \exp(\theta)$ *we have* $\psi^*(a) = a\log(a) - a$*.*

Bregman distance regularization as a log-maximum a-posteriori estimator for an exponential family.

The probability (normally and identically distributed) of observing $u_i^\delta := u^\delta(x_i)$ given $u_i := F(a)(x_i)$ is given by

$$p(u_i^\delta | u_i) = \frac{1}{\varpi\sqrt{2\pi}} \exp\left(-\frac{|u_i^\delta - u_i|^2}{2\varpi^2}\right) .$$

where $a_i = a(x_i) \in \mathbb{R}$.

According to Theorem 7

$$p(a) = \exp\left(-D_{\psi^*}(\hat{a}, a)\right)\exp(\psi^*(\hat{a}))p_0(\hat{a}) \;.$$

The Log-maximum estimation then consists in minimizing the functional

$$\vec{a} \longmapsto \sum_i \Big(-\log(p(u_i^\delta | u_i)) - \log(p(a_i))\Big) \,,$$

which is equivalent to minimizing the functional

$$\vec{a} \longmapsto \sum_i (u_i - u_i^\delta)^2 + \beta \sum_i D_{\psi^*}(\hat{a}_i, a_i) \,,$$

where $\beta = 2\varpi^2$.

Example

Exponential family associated to Poisson distributions, consisting in minimization of

$$a \longmapsto \mathcal{F}_{\beta,u^\delta}(a) := \|F(a) - u^\delta\|^2_{L^2(\Omega)} + \beta KL(\hat{a}, a) \ , \qquad (23)$$

where

$$KL(\hat{a}, a) = \int_\Omega a \log(\hat{a}/a) - (\hat{a} - a)\, dx \ \ .$$

Lemma

$\Omega \subset \mathbb{R}^2$ bounded and $F : L^1(\Omega) \longrightarrow L^2(\Omega)$ is continuous with respect to the weak topologies, respectively.

1. *Let $a, b \in \mathcal{D}(\mathcal{G})$. Then*

$$\|a - b\|^2_{L^1(\Omega)} \leq \left(\frac{2}{3}\|a\|_{L^1(\Omega)} + \frac{4}{3}\|b\|_{L^1(\Omega)} \right) KL(a,b) \,. \tag{24}$$

2. *For sequences $(a_k)_k$ and $(b_k)_k$ in $L^1(\Omega)$, such that one of them is bounded: If $KL(a_k, b_k) \to 0$, then $\|a_k - b_k\|_{L^1(\Omega)} \to 0$.*
3. *Let $0 \neq \hat{a} \in \mathcal{D}_B(\mathcal{G})$, $\mathcal{M}_{\beta,u^\delta}(M) := \{a \in \mathcal{D}_B(\mathcal{G}) : \mathcal{F}_{\beta,u^\delta}(a) \leq M\}$ are weak sequentially compact.*

Convergence analysis

Using standard results on variational regularization, we have:

Theorem

There exists a minimizer of $\mathcal{F}_{\beta,u^\delta}$ *in* (23)*. The minimizers are stable and convergent for* $\beta(\delta) \to 0$ *and* $\delta^2/\beta(\delta) \to 0$*. Stable means that* argmin $\mathcal{F}_{\beta,u^{\delta_k}} \to$ argmin $\mathcal{F}_{\beta,u^0}$ *for* $\delta_k \to 0$ *and that* argmin $\mathcal{F}_{\beta(\delta_k),u^{\delta_k}}$ *converges to a solution of* (12) *with minimal energy.*

Conclusions

- The main novelty

i. f only requires convexity properties and weak lower-semicontinuity.

ii. we establish, for Bregman distances, better convergence rates than those available in the literature to this problem,

iii. Another advantage of the current approach is the requirement of weaker conditions than those previously required in the literature. Namely, we only require (20).

iv. we prove (19),

v. and we motivate Bregman distance regularization using exponential families.

Future research

- Future research

i numerical implementation.

ii investigation of American Options.

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