Uniqueness and regularization for unknown spacewise lower-order coefficient and source for the heat type equation

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Abstract:

In this contribution we show sufficient conditions for simultaneous unique identification of unknown spacewise coefficients and heat source in a parabolic partial differential equation given additional final time measurements. Our approach is based on density, in suitable spaces, of the corresponding adjoint problem.

A second issue of this paper is the regularization approach. The sequence of approximated solution is obtained by coupling the nonlinear Landweber iteration with iterated Tikhonov regularization. We show that the parameter-to-solution map satisfies sufficient conditions to prove stability and convergence of approximated solutions for the identification problem. We use a unified discrepancy principle as the stopping criteria.

Finally, we apply the developed theory in the inverse identification problem of unknown parameters (perfusion coefficient, metabolic heat source) for the identification of tumor regions by thermography.

Keywords: uniqueness, thermophysical parameters and source identification, iterative regularization, parabolic type equation, final time measurements.

1 Introduction

Coefficients identification inverse problems have the characteristic of being ill-posed [4]. In other words, typically, solutions for such problems may fail to exist, may not be unique or be unstable under errors in the input data. The issue of existence can be relaxed by considering generalized solutions. On the other hand, uniqueness and stability are crucial for obtaining a reasonable solution for the coefficient identification inverse problem, theoretically as well as in terms of numerical approximations.

Uniqueness in inverse problems have been studied for a long time. In particular, for coefficient identification in parabolic equations, see [26, 8] and references therein. However, only recently some attention has been given to the uniqueness of coefficient identification in parabolic type equation with final time measurements [2, 5, 8, 26]. A method that became very popular recently to prove uniqueness and conditional stability in coefficient identification inverse problems are Carleman type estimates, e.g. [8, 13, 26] and references therein. In particular in [26] there is a complete overview of Carleman estimates for parameter identification of inverse parabolic problems. Recently, in [2],

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the authors proposed using Carleman type estimates for identification of spacewise source and heat conductivity in a heat type equation for a given additional final time measurement. For the one-dimensional heat equation, in [2], density properties of the associated adjoint problem, in suitable spaces and an additional assumption on the final measurements was used to guarantee uniqueness identification of spacewise source and heat conductivity. The duality type method was used before in [6, 5] for proving uniqueness identification of spacewise coefficients in parabolic type equations. However, in [6, 5] the author does not consider simultaneous identification of coefficients and heat source.

Likewise uniqueness, regularization approaches for parameter identification in parabolic partial differential equations also have a long history and a full overview becomes almost impossible. See, for example, [15, 9, 4, 8] and references therein. We remark that, in general, parameter identification inverse problems are nonlinear, even if the forward problem is linear. Therefore, to prove convergence and stability of iterative regularization methods in this context, ones need to prove some nonlinearities conditions for the parameter-to-solution map [12], that are, in general, hard to be verified in practice.

Summarizing, the main contributions of this paper are: We first use the density of the associated adjoint problem to prove sufficient conditions for the simultaneous uniqueness identification of spacewise heat source and coefficients that multiply the lower order term (or lower-order derivative, see Remark 2) of a parabolic type equation, with some assumption in the given final data (see equation (3) below). It is worth noting that we do not have restrictions of space dimension, except that necessaries to prove existence and uniqueness of a solution of partial differential equation. Such restrictions are related to the smoothness of the initial and boundary conditions and the smoothness of coefficients of the partial differential equation.

We also propose a iterative regularization method that consists in to couple Landweber and iterated Tikhonov regularization strategies. We prove properties of the parameter-to-solution maps (see definitions (6) - (8)) that are sufficient conditions to show convergence and stability of regularized solutions with respect to the noise level. We use a stopping criteria given by a unified discrepancy principle. Such discrepancy principle has the characteristic of reduce computational effort (See Remark 3).

Finally, we apply the theory developed before to prove uniqueness and provide and regularization approach for the identification of the blood perfusion rate and the metabolic heat generation in a thermography application for melanoma diagnoses.

The paper is organized as follows: In the remaining part of this section we introduce some notations. In Section 2, we present the model problem and define the parameter-to-solution map associated with the identification problem. In Section 3, we prove the uniqueness identification of the spacewise pair of coefficient and source from the additional final time measurement in the parabolic partial differential equation model. In Section 4, we prove properties of the parameter-to-solution map that guarantees the convergence and stability of the iterative regularized solutions, that will be proposed in Section 5. In Section 6, we present an application of the theory developed early in a melanoma diagnoses from thermography. Finally, in Section 7, we present some conclusions and future works.

Notations: By $L^p(\Omega)$ for $1 \le p < \infty$, we denote the usual space of p-integrable functions on Ω with the usual norm $\|\cdot\|_{L^p(\Omega)}$. The space $L^\infty(\Omega)$ is the standard L^∞-space. We denote by $W^{k,p}(\Omega)$ the standard Sobolev space on Ω with generalized derivatives of order $\le k$ in $L^p(\Omega)$. In particular, for $p = 2$ we have the Hilbert spaces $H^k(\Omega)$.

Let $T > 0$ be fixed. Define the measurable function $u(\cdot, t) : (0, T) \longrightarrow X$, where X is a Banach space. We denote by $C([0, T]; X)$ the space of continuous mappings $u(\cdot, t)$ with the usual norm and by $L^2((0, T); X)$ the space of functions such that

$$\int_0^T \|u(\cdot, t)\|_X^2 dt < \infty .$$

2 Model Problem

In this paper, we are considering a thermal-physical model in a non-homogeneous and non-isotropic body, denoted by Ω occupying an open, bounded and smooth domain in $\mathbb{R}^n$, described by the parabolic type partial differential equation

$$\begin{aligned} u_t - L(a, b, c)u = f(x) \text{ in } \ \Omega \times (0, T) \\ u(x, t) = 0 \text{ for } (x, t) \ \in \partial\Omega \times (0, T) \\ u(x, 0) = \varphi(x) \text{ for } x \in \Omega , \end{aligned} \tag{1}$$

for a time interval $(0, T)$ with $T > 0$, where

$$L(a, b, c)u = \nabla \cdot (a(x)\nabla u) - b(x) \cdot \nabla u - c(x)u , \tag{2}$$

is a linear elliptic differential operator of second order with all the coefficients time independent. Moreover, a, c are strictly positive real valued function in $L^\infty(\Omega)$ with $0 < \underline{a} \le a(x)$ and $0 < \underline{c} \le c(x)$, for $x \in \Omega$, b is a real valued vector function sufficiently smooth. $f \in L^2(\Omega)$ is the spacewise heat source. For simplicity, we assume that the given initial temperature distribution $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$, and that $b = 0$. Hence, we can assume that information that u is identically zero on the boundary of Ω. We will denote $L(a, b, c) = L(a, c)$.

Moreover, we assume given the additional final temperature measurement $g \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying

$$u(x, T) = g(x) \text{ for } x \in \Omega , \quad T > 0 . \tag{3}$$

Since the parameters in (1) are sufficient smooth, we can, formally, define the adjoint of the partial differential operator $L(a, c)$ as

$$L(a, c)^* v = \nabla \cdot (a(x)\nabla v) + \nabla(b(x)v) - c(x)v . \tag{4}$$

Given the assumptions on the parameters in the model (1), follows from the Hille-Yosida Theorem [27] that the operator $-L$ generate a strictly dissipative and contraction C_0 semigroup $\{G(t)\}_{t\in\mathbb{R}^+}$ in $L^2(\Omega)$ with $\mathcal{D}(L) = \{u : u \in H_0^1(\Omega) , Lu \in L^2(\Omega)\}$. Hence, $\|G(T)\| < 1$. Note that, since $H_0^1(\Omega)$ is compact embedded in $L^2(\Omega)$, $G(t)$ is a compact operator for every $t > 0$. Moreover, it follows from classical results on parabolic partial differential equations, e.g. [16, 14] that there exists a unique solution $u \in C^1((0,T), H_0^1(\Omega) \cap H^2(\Omega))$ of (1)-(3) with

$$\|u\|_1 := \int_0^T \left(\|u_t(\cdot,t)\|^2_{L^2(\Omega)} + \|u(\cdot,t)\|^2_{H^2(\Omega)} \right) dt < \infty. \tag{5}$$

Remark 1. *i) Is possible to extend the result to a weaker assumption on the data, but one then has to consider appropriate weak formulation of* (1)-(3)*, [14].*

ii) Since we have time-independent coefficients and source, it follows that the solution u to (1) *with additional data* (3) *is analytic in time. This means that u has derivatives of all orders with respect to t [8, 25]. Moreover u is at least continuous with respect to time, and, therefore, we can conclude that pointwise evaluation in time makes sense.*

The inverse problem that we are interested here is recover the pair of spacewise parameter and source $(c(x), f(x))$ in (1) from the additional final time measurement (3).

Assuming that $b = b(x)$ is not zero in (1), one can recover the pair of spacewise parameter and source $(b(x), f(x))$ in (1) from the additional final time measurement (3). However, the uniqueness result, in this case, follows from a very similar argument (See Remark 2) developed with this contribution and we will not fix in this problem here.

2.1 The parameter-to-solution map

In this subsection, we introduce the parameter-to-solution map associated with the parameter identification problem discussed previously. For now, we will consider the following admissible set of spacewise coefficient and heat source:

Definition 1. *We denote the admissible set of functions as*

$$\mathcal{D}(\mathbb{F}) := \{(c,f) \in L^\infty(\Omega) \times L^2(\Omega) \;\; s.t. \;\; 0 < \underline{c} \leq c(x) \leq \overline{c}\,, a.e. \;\; in\; \Omega\}\,.$$

Moreover we denote by $\mathcal{D}_c(\mathbb{F}) \subset L^\infty(\Omega)$ and $\mathcal{D}_f(\mathbb{F}) \subset L^2(\Omega)$ the restriction of $\mathcal{D}(\mathbb{F})$ to the first and second component of the pair (c,f), respectively.

Note that, since Ω is bounded, $\mathcal{D}(\mathbb{F})$ is a convex and closed subset of $L^2(\Omega) \times L^2(\Omega)$. However, since $L^\infty(\Omega)$ can not be continuously embedding in $H^1(\Omega)$ for $\Omega \subset \mathbb{R}^2$, $\mathcal{D}(\mathbb{F})$ has no interior points when equipped with the $H^1(\Omega)$ norm. This will not affect the convergence analysis that follows. See Section 4 for details.

Moreover, let $u \in C([0,T]; H_0^1(\Omega) \cap H^2(\Omega))$ be the unique solution of (1), with $(c,f) \in \mathcal{D}(\mathbb{F})$. Then, it follows that the restriction $u(x,t_0)$ is well-defined for $0 \leq t_0 \leq T$. Therefore, the restriction of u to $\Omega \times \{T\}$ exists, it is, $u(x,T) = g(x)$ is well defined and, moreover, the nonlinear operators

$$\begin{aligned} \mathbb{F} : \mathcal{D}(\mathbb{F}) \subset L^2(\Omega) \times L^2(\Omega) &\longrightarrow L^2(\Omega) \\ (\,c\,,\,f\,) \quad &\longmapsto \quad \mathbb{F}(c,f) = g(x) \end{aligned} \tag{6}$$

is well-defined.

A second operator equation that we are interested in is the restriction of the operator $\mathbb{F}$ defined in (6) in $\mathcal{D}_f(\mathbb{F})$. Indeed, it introduces a family of operators, parameterized by $c \in \mathcal{D}(\mathbb{F})$ defined by

$$\begin{aligned} F_c \ : \mathcal{D}_f(A) \subset L^2(\Omega) &\longrightarrow L^2(\Omega) \\ f &\longmapsto F_c(f) := \mathbb{F}_{|_{\mathcal{D}_f(\mathbb{F})}}(c, f) = g(x)\,. \end{aligned} \tag{7}$$

Note that, for any fixed $c \in \mathcal{D}_f(\mathbb{F})$, the operator F_c is nonlinear, unless $\varphi = 0$.

Finally, we introduce the restriction of the operator $\mathbb{F}$ to $\mathcal{D}_c(\mathbb{F})$ define a family of nonlinear operators, parameterized by $f \in L^2(\Omega)$, defined by

$$\begin{aligned} A_f \ : \mathcal{D}_c(\mathbb{F}) \subset L^2(\Omega) &\longrightarrow L^2(\Omega) \\ c &\longmapsto A_f(c) := \mathbb{F}_{|_{\mathcal{D}_c(\mathbb{F})}}(c, f) = g(x)\,. \end{aligned} \tag{8}$$

In practical applications the final temperature (3) is, in general, not known exactly. One is given only approximate measured data $g^\delta \in L^2(\Omega)$, corrupted by a noise level $\delta > 0$ satisfying

$$\left\| g - g^\delta \right\|_{L^2(\Omega)} \leq \delta\,. \tag{9}$$

Therefore, given the measurements g^δ, the inverse problem is: Find $(c(x), f(x)) \in \mathcal{D}(\mathbb{F})$ such that

$$\mathbb{F}(c, f) = g^\delta\,, \qquad \text{for } g^\delta \text{ satisfying (9)}\,. \tag{10}$$

The inverse problem is ill-posed in the sense of Hadamard [4]. In other words the solution of the inverse problem is unstable with respect to noise data, it is, small perturbation in the given data implies in large perturbation on the parameter space. Examples of instability of reconstruction the coefficient c can be obtained in [6]. For instabilities examples in the reconstruction of f see [10, 22]. Hence, beyond uniqueness, some regularization methods need to be used to guarantee stability of the parameter and source reconstructions, given a set of noisy data.

3 Uniqueness for the spacewise coefficient and source

In this section, we wish to prove uniqueness of the spacewise coefficient c and source f in (1) by additional measurement (3). We will use a similar approach of [2, Section 3] for the identification of the heat conductivity and heat source for the one-dimensional version of the heat equation (1). The derivation of the uniqueness result is based on a completely different technique than Carleman estimates [2, 8, 26]. Indeed, the technique is based on results of density, in certain function spaces, of solutions of the corresponding adjoint problem [6, 5] and the unique continuation principle [16, 14]. Moreover, the proposed approach is different of the maximum principle used in [8, Section 9.1]. It is worth remark that, differently of [8, 6, 26], we wish to show uniqueness of the spacewise coefficient c and source f, simultaneous, from (1)-(3).

The steps for proving uniqueness of the identification of the pair of parameters $\{c(x), f(x)\}$ for giving initial and final data in (1)-(3) are outlined as follows:

Preliminary results: Denote by $u = u(c_1, f_1)$ and $\upsilon = u(c_2, f_2)$ the respective solutions of (1) with additional data (3), for coefficient and source satisfying the Definition 1. Then, for linearity of (1) the difference $w = u - \upsilon$ satisfies

$$w_t - L(a, c_1)w = (c_1(x) - c_2(x))\upsilon + (f_1(x) - f_2(x)) \text{ in } \Omega \times (0, T) \tag{11}$$

with homogeneous initial, boundary and final conditions.

Let we invoke the adjoint problem of (11), that reads as

$$\begin{aligned} \psi_t + L^*(a, c_1)\psi = 0 \text{ in } \Omega \times (0, T) \\ \psi(x, t) = 0 \text{ for } (x, t) \in \partial\Omega \times (0, T) \\ \psi(x, 0) = 0 \text{ for } x \in \Omega\,, \\ \psi(x, T) = \mu(x) \text{ for } x \in \Omega,\ , \end{aligned} \tag{12}$$

where $\mu(x)$ is an arbitrary function in $C_0^2(\overline{\Omega})$.

For properties of solutions of the adjoint equation (12) we have:

Lemma 1. *Let the Assumption on the coefficients, source, initial, boundary and final conditions in (1)-(3) hold. Then:*

i) For any function $\mu(x) \in C_0^2(\overline{\Omega})$, there exists a unique solution $\psi(x, t; \upsilon) \in C^1((0, T); H_0^1(\Omega) \cap H^2(\Omega))$ of (12).

ii) For any function $\mu(x) \in C_0^2(\overline{\Omega})$, the following relation holds

$$\int_0^T \int_\Omega \psi(x, t; \mu(x))F(x, t)dxdt = 0, \tag{13}$$

where w is a solution to (11) with right-hand side $F(x, t) := (c_1(x) - c_2(x))\upsilon + (f_1(x) - f_2(x))$.

iii) For $\mu(x)$ ranging over the space $C_0^2(\overline{\Omega})$, the corresponding range of $\psi(x, t; \mu(x))|_{t=\tau}$ is everywhere dense in $L^2(\Omega)$ at any time $t = \tau$, $0 \le \tau \le T$.

iv) Given that

$$\int_0^T \int_\Omega \psi(x, t; \mu(x))\Phi(x, t)dxdt = 0$$

for $\mu(x)$ ranging over the space $C_0^2(\overline{\Omega})$, then $\Phi(x, T) = 0\,,$ a.e. $x \in \Omega\,.$

Proof. Item i) is a well-known result for parabolic equations, see, for example, [16, 14].
Item ii) follows immediately by multiplication of (15) by ψ and integration by parts.
Item iii) and Item iv) are consequences of [5, Lemma 2-3, pg 318] applied for a multi-dimensional case. □

Lemma 2. *Let w satisfying (11). Then $w(x, t) = 0$ almost everywhere for $(x, t) \in \overline{\Omega} \times [0, T]$.*

Proof. Note that, since w has homogeneous boundary conditions, it follows for standard parabolic theory that $w(\cdot,t) \in H_0^1(\Omega)$, for any $t \in (0,T)$. From Remark 1 ii) we have that w has derivatives of all orders with respect to t and time pointwise evaluation makes sense.

Applying Lemma 1 Item ii) in combination with iv) in (11) we get

$$[(c_1(x) - c_2(x))v(x,T) + (f_1(x) + f_2(x))] = 0\,, \quad \text{a.e. } x \in \Omega\,. \tag{14}$$

Using this in the first equation in (11) in combination with $w(x,T) = 0$ for a.e. x in Ω, we conclude that $w_t(x,T) = 0$, a.e. x in Ω.

Define $z = w_t$. Since coefficients and source are time independent, we have that z satisfies

$$\begin{aligned} z_t - L(a,b,c_1)z &= (c_1(x) - c_2(x))v_t \text{ in } \ \Omega \times (0,T) \\ z(x,t) &= 0 \text{ for } (x,t) \in \partial\Omega \times (0,T) \\ z(x,0) &= \theta_1(x) \text{ for } x \in \Omega\,, \\ z(x,T) &= 0 \text{ for } x \in \Omega. \end{aligned} \tag{15}$$

Splitting this problem into two, one with zero right-hand side and with initial condition θ_1 and one with the given right-hand side and zero initial condition, following the proof of [2, Lemma 2] one can conclude that the solution to the first one is identically zero, i.e. $\theta_1(x) = 0$. Therefore, since z satisfies a problem of the same kind as w we can again apply Lemma 1 Item ii) in combination with iv) in (15) to conclude that

$$(c_1(x) - c_2(x))v_t(x,T) = 0\,, \quad \text{a.e. } x \in \Omega\,. \tag{16}$$

Using this in the first equation in (15) in combination with $z(x,T) = 0$ for a.e. x in Ω, we conclude that $z_t(x,T) = 0$, i.e. $w_{tt}(x,T) = 0$, a.e. x in Ω. Continuing this, putting $z_1 = z_t$ and deriving the problem for z_1 and applying the similar reasoning, i.e. Lemma 1 Item ii) in combination with iv), we find that $w_{ttt}(x,T) = 0$. Further continuing this it is possible to prove that $(\partial_t^{(k)} w)(x,T) = 0$ for $k = 0, 1, 2, \ldots$. From this and strong unique continuation results for parabolic equations [16, 14], we conclude that $w(x,t) = 0$ for a.e. $(x,t) \in \Omega \times [0,T]$. □

The uniqueness proof: We now have the required results in order to prove the main step in the uniqueness of $(c(x), f(x))$ in (1), with additional final data (3).

Theorem 3. *Let the Assumption on this paper holds. Moreover, assume that $g(x) \neq \varphi(x)$ for a.e $x \in \Omega$. Then the inverse problem (1)-(3) has a unique solution $\{u, c, f\}$ with the coefficient $c \in L^\infty(\Omega)$, the heat source $f \in L^2(\Omega)$, and temperature u, with $\|u\|_1 < \infty$.*

Proof. Follows from Lemma 2 that w is identically zero in $\Omega \times [0,T]$. This in particular implies that $z = 0$ in $\Omega \times [0,T]$.

From the first equation in (15) we then have

$$(c_1(x) - c_2(x))v_t(x,t) = 0\,, \text{for a.e. } (x,t) \in \Omega \times [0,T]\,.$$

Since the coefficients are independent of time, we integrating with respect to time, form 0 to T and use the fundamental theorem of calculus, to get

$$(c_1(x) - c_2(x))(g(x) - \varphi(x)) = (c_1(x) - c_2(x))(v(x,T) - v(x,0)) = 0\,, \text{for a.e. } x \in \Omega\,.$$

From assumptions on g and φ and we can conclude that $c_1(x) = c_2(x)$ also for a.e $x \in \Omega$.

Moreover, since $w = 0$ (from Lemma 2) and $c_1(x) = c_2(x)$, we have from (11) that

$$f_1(x) - f_2(x) = 0\,, \text{for a.e. } x \in \Omega\,.$$

The uniqueness of u, with $\|u\|_1$ follows from the standard theory of solution of parabolic partial differential equations [14]. □

It is worth to note that the argument in the proof of Theorem 3 goes beyond the proof of uniqueness in [2]. The main reason is that, for the lower order terms we do not have the influence of the divergent operator. Therefore the proof is still true in multidimensional heat equations.

Remark 2. *With similar argumentation of Theorem (3) is possible to prove uniqueness of $\{u, b, f\}$ for the model (1)-(3). Indeed, some small modifications in the derivations of the steps above are necessary. The main difference is that in Theorem 3 we need the assumption that $|\nabla g(x) - \nabla \varphi(x)| > 0$ for a.e $x \in \Omega$.*

4 Properties of the Parameter-to-Solution Map

Before introducing the iterative regularization in Section 5, we need to prove some properties of parameter-to-solution map defined before allowing us to obtain convergence, stability and regularization properties of approximated solutions.

4.1 Continuity

The first result in this direction is the continuity of operators $\mathbb{F}$, A_f and F_c defined in (6) - (7), respectively.

Theorem 4. *The operator $\mathbb{F} : \mathcal{D}(\mathbb{F}) \subset L^2(\Omega) \times L^2(\Omega) \longrightarrow L^2(\Omega)$, defined in (6) is continuous.*

Proof. Let $\{(c_n, f_n)\}$ be a sequence in $\mathcal{D}(\mathbb{F})$ converging to some $(c_0, f_0) \in \mathcal{D}(\mathbb{F})$ w.r.t. $L^2(\Omega)\times L^2(\Omega)$-norm. Denote by $u_n = u(c_n, f_n)$ and $v = u(c_0, f_0)$, respectively, the solutions of (1). As before, the difference $w := u_n - v$ satisfies

$$w_t - \nabla \cdot (a(x)\nabla w) + c_n(x)w = (c_0 - c_n)v + (f_n - f_0)$$

with homogeneous, initial, boundary and final conditions.

Since $u_n, v \in L^2(0,T; H_0^1(\Omega)) \cap C([0,T]; H_0^1(\Omega))$, for each $t \in [0,T]$ we have

$$\int_\Omega w_t w - \nabla \cdot (a_n \nabla w)w + c_n(x)wwdx = \int_\Omega (c_0 - c_n)vwdx + \int_\Omega (f_n - f_0)wdx$$

The Green's formula [16], implies that

$$\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \underline{a}\|\nabla w(t)\|^2_{L^2(\Omega)} + \underline{c}\|w(t)\|^2_{L^2(\Omega)} \leq \int_\Omega |c_n - c_0||v(t)||w(t)|dx + \int_\Omega |f_n - f_0||w(t)|dx\,,$$

where we used the homogenous initial and boundary conditions of the equation satisfied by w. Moreover, from Theorem A.2, there exists some $\tilde{\varepsilon} > 0$ such that $\|v(t)\|_{W^{1,q}(\Omega)} \leq C$, for $q = 2 + \tilde{\varepsilon}$. Since the application $t \mapsto \|v(t)\|_{W^{1,q}(\Omega)}$ is continuous ($v \in C([0,T], W^{1,q}(\Omega))$), we have that is uniformly bounded for $t \in [0,T]$.

Let $p^{-1} + q^{-1} = 2^{-1}$. Using the Hölder inequality with $p^{-1} + q^{-1} + 2^{-1} = 1$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} &+ \underline{a}\|\nabla w(t)\|^2_{L^2(\Omega)} + \underline{c}\|w(t)\|^2_{L^2(\Omega)} \\
&\leq \|v(t)\|_{W^{1,q}(\Omega)}\|c_n - c_0\|_{L^p(\Omega)}\|w(t)\|_{L^2(\Omega)} + \|f_n - f_0\|_{L^2(\Omega)}\|w(t)\|_{L^2(\Omega)} \\
&\leq C\left(\|c_n - c_0\|_{L^p(\Omega)} + \|f_n - f_0\|_{L^2(\Omega)}\right)\|w(t)\|_{L^2(\Omega)}.
\end{aligned}$$

Since, for each $t \in (0,T)$ $w \in H^1_0(\Omega)$, it follows from the Poincaré inequality that $\|w\|^2_{L^2(\Omega)} \leq C_1\|\nabla w\|^2_{L^2(\Omega)}$. This, together with the Young inequality with $\hat{\varepsilon}$ [16], yields

$$\begin{aligned}
min\{2^{-1}, C_1\underline{a}, \underline{c}\}&\left(\frac{d}{dt}\|w(t)\|^2_{L^2(\Omega)} + \|w(t)\|^2_{L^2(\Omega)}\right) \\
&\leq \frac{C}{\hat{\varepsilon}}\left(\|c_n - c_0\|_{L^p(\Omega)} + \|f_n - f_0\|_{L^2(\Omega)}\right)^2 + C\hat{\varepsilon}\|w(t)\|^2_{L^2(\Omega)}
\end{aligned}$$

Let $\hat{\varepsilon} > 0$ such that $C\hat{\varepsilon} > \min\{2^{-1}, C_1\underline{a}, \underline{c}\}$. Using the Gronwall inequality, it follows that

$$\|w(t)\|_{L^2(\Omega)} \leq \underline{C}\left(\|c_n - c_0\|_{L^p(\Omega)} + \|f_n - f_0\|_{L^2(\Omega)}\right)^2 e^{ct}$$

for all $t \in [0,T]$. Given the continuity of the solution of (1) with respect to t the inequality holds for $t = T$.

Now the conclusion follows form Lemma A.1 in Appendix. □

As a corollary of Theorem 4 we conclude the continuity of operators A_f and F_c, for any fixed f and c in $\mathcal{D}(\mathbb{F})$, respectively.

Corollary 5. *For each fixed $c \in \mathcal{D}(\mathbb{F})$, the operator F_c defined in (7) is continuous in $L^2(\Omega)$. For each fixed $f \in \mathcal{D}(\mathbb{F})$, the operator A_f defined in (8) is continuous in $L^2(\Omega)$.*

4.2 Fréchet Derivative and Tangential Cone Condition

An important result to guarantee convergence of iterative regularization methods for nonlinear inverse problems is the local tangential cone condition. We will prove such properties in the next two propositions.

Proposition 6. *For each fixed $c \in \mathcal{D}_c(\mathbb{F})$ the operator F_c is Fréchet differentiable. The Fréchet derivative is Lipschitz continuous and satisfies the local tangential cone condition. In other words, for each $f, \tilde{f} \in B_\rho(f_0) \subset D_f(\mathbb{F})$, there exists a $0 < \eta < 1$, such that*

$$\left\|F_c(\tilde{f}) - F_c(f) - F_c'(f)(\tilde{f} - f)\right\|_{L^2(\Omega)} \leq \eta\left\|F_c(\tilde{f}) - F_c(f)\right\|_{L^2(\Omega)}. \tag{17}$$

Proof. Let $h \in L^2(\Omega)$. By linearity of (1), the sensitivity $u' \cdot h = u(a, f + h) - u(f)$ satisfies

$$(u' \cdot h)_t - \nabla \cdot (a\nabla(u' \cdot h)) + c(x)u' \cdot h = h\,, \tag{18}$$

with homogeneous initial, final and boundary condition. Follows from the standard parabolic theory that there exists a unique solution $C([0,T], H_0^1(\Omega) \times H^2(\Omega))$ of (18) and

$$\|u' \cdot h\|_{C([0,T],H^1(\Omega))} \leq C\|h\|_{L^2(\Omega)}\,.$$

Let $\{G(t)\}_{t\in\mathbb{R}^+}$ be the semigroup generated by the differential operator $-L(a,c)$. Define the linear operator $K(t) : L^2(\Omega) \longrightarrow L^2((0,T); H_0^1(\Omega)) \cap C([0,T]; H_0^1(\Omega))$ by $K(t)f = \int_0^t G(t-s)f ds$. Note that the solution u of (1) is formally given by $u(x,t) = G(t)g(x) + K(t)f(x)$. Therefore, solving the operator equation (7) is equivalent to solving

$$K(T)f = u(c, x, T) - G(T)g(x)\,.$$

In other words, $(F_c'(f))(h) = K(T)h = u'(x,T) \cdot h$, where $u' \cdot h$ is the unique solution of (18). Therefore, follows from the linearity and continuity of K that F_c' is Lipschitz continuous and satisfies the tangential cone condition. ☐

Proposition 7. *For each fixed $f \in \mathcal{D}_f(\mathbb{F})$, the operator A_f is differentiable in the direction κ such that $c + \kappa \in D_c(\mathbb{F})$. The derivative can be continuously extended as a linear operator to $H^1(\Omega)$. The extension is Lipschitz continuous. Moreover, the local tangential cone condition is satisfied. In other words, there exists $\rho > 0$ and $0 < \eta < 1$ such that for each $c, \tilde{c} \in B_\rho(c_0) \subset D_c(\mathbb{F})$,*

$$\left\|A_f(\tilde{c}) - A_f(c) - A_f'(c)(\tilde{c} - c)\right\|_{L^2(\Omega)} \leq \eta\|A_f(\tilde{c}) - A_f(c)\|_{L^2(\Omega)}\,. \tag{19}$$

Proof. Let $\kappa \in D_c(\mathbb{F})$. Then, by linearity and continuity with respect to the coefficients of solutions of equation (1) we have that the directional derivative $u'_{(c)} \cdot (\kappa)$ in the direction κ such that $c + \kappa \in \mathcal{D}(\mathbb{F})$ satisfies

$$(u'_{(c)} \cdot (\kappa))_t - L(a,c)u'_{(c)} \cdot (\kappa) = \kappa u\,, \tag{20}$$

with homogeneous initial and boundary conditions. It follows from standard parabolic partial differential equation theory [16, 14] that there exists a unique solution $u'_{(c)} \cdot (\kappa) \in C^1((0,T); H_0^1(\Omega) \cap H^2(\Omega)) \cap C^0([0,T]; L^2(\Omega))$ of (20). Hence, the $u'_{(c)} \cdot (\kappa)(t)$ make sense, for every $t \in [0,T]$. Moreover,

$$\left\|u'_{(c)} \cdot (\kappa)\right\|_{C^0([0,T],L^2(\Omega))} \leq \|\kappa u\|_{L^2(\Omega\times(0,T))}\,.$$

Since κ is time-independent, follows from the Cauchy-Schwarz inequality that

$$\left\|u'_{(c)} \cdot (\kappa)(T)\right\|_{L^2(\Omega)} \leq C(T)\|\kappa\|_{H^1(\Omega)}\|u\|_1\,.$$

Therefore, the directional derivative $u'_{(c)} \cdot (\kappa)$ can be extended to $H^1(\Omega)$ as a bounded linear operator.

To prove the Lipschitz continuity, let $c, \tilde{c} \in \mathcal{D}(\mathbb{F})$ and $u = u(c), \tilde{u} = u(\tilde{c})$ the respective solution of equation (1). Then the difference $w = u'_{(c)} \cdot (\kappa) - \tilde{u}'_{(\tilde{c})} \cdot (\kappa)$ satisfies

$$w_t - L(a,c)w = (c - \tilde{c})\tilde{u} + \kappa(u - \tilde{u})\,.$$

As before, we have

$$\|w(T)\|_{L^2(\Omega)} \leq \|c-\tilde{c}\|_{H^1(\Omega)}\|u\|_1 + \|\kappa\|_{H^1(\Omega)}\|u-\tilde{u}\|_{L^2((0,T)\times\Omega)} \,.$$

With the same argumentations as in Theorem 4, we have $\|u-\tilde{u}\|_{L^2((0,T)\times\Omega)} \leq C\|c-\tilde{c}\|_{H^1(\Omega)}$ and the Lipschitz continuity follows.

Moreover, from the linearity of equations (1) and (20) we get that $v = u(c)-u(\tilde{c})-u'(\tilde{c})\cdot(c-\tilde{c})$ satisfies

$$v_t - L(a,\tilde{c})v = (c-\tilde{c})(u(c)-u(\tilde{c}))\,, \tag{21}$$

with homogeneous boundary and initial conditions. Using similar argumentation as in the proof of Theorem 4, we obtain that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{L^2(\Omega)} + \underline{a}\|\nabla v(t)\|^2_{L^2(\Omega)} + \underline{c}\|v(t)\|^2_{L^2(\Omega)} &\leq \int_\Omega (c-\tilde{c})(u(c)-u(\tilde{c}))v(t)dx \\ &\leq \frac{\|c-\tilde{c}\|_{L^\infty(\Omega)}}{\varepsilon}\|u(c)-u(\tilde{c})\|^2_{L^2(\Omega)} + \varepsilon\|v(t)\|^2_{L^2(\Omega)}\,,\end{aligned}$$

where we use the Young inequality with ε.

Let we take $\varepsilon < \underline{c}/2$. Therefore, we have

$$\|v(t)\|_{L^2(\Omega)} \leq C\|u(c)-u(\tilde{c})\|_{L^2(\Omega)}\,, \quad \forall t \in (0,T)$$

where $C = C(\|c-\tilde{c})\|$ and some constants that are independent of the solution of (1). Denoting $\eta := C(\|c-\tilde{c}\|)$, we have that

$$\|v(t)\|_{L^2(\Omega)} \leq \eta\|u(c)-u(\tilde{c})\|_{L^2(\Omega)}\,, \quad \forall t \in (0,T)\,.$$

Given the continuity of v and u with respect to t, the inequality holds for $t = T$. □

Adjoint of the Fréchet derivative: Let we finish this section making the calculation of the adjoint of the Fréchet derivative of the operator defined in (7).

Lemma 8. *Let $r_f \in L^2(\Omega)$. Then the adjoint of the Fréchet derivative $F'_c(f)$ denoted by $(F'_c(f))^* : L^2(\Omega) \to L^2(\Omega)$ is given by*

$$(F'_c(f))^*(r_f) = -V(x,0)\,, \tag{22}$$

where $V \in C([0,T], H^1_0(\Omega))$ is the unique solution of

$$\begin{aligned} V_t + \nabla\cdot(a\nabla V) + cV = r_f\,, &\quad \text{in } \ (0,T)\times\Omega\,, \\ V(x,t) = 0\,, &\quad (x,t) \in \partial\Omega\times(0,T) \\ V(x,T) = 0\,, &\quad \text{for } x \in \Omega\,. \end{aligned} \tag{23}$$

Proof. Existence, uniqueness and regularity of the solution of (23) follows from standard parabolic theory [14]. Note that equation (23) is the adjoint equation of (18). Therefore, the assertion follows directly from integration by parts. See the details of the calculations in [10]. □

5 An Iterative Regularization Method

In this section, we propose an iterative regularization method to regularize the solution of the inverse problem (6). It consist in a coupled Landweber - iterated Tikhonov regularization approach given by the iteration

$$
\begin{aligned}
&\text{Given } c_0 = c_0^\delta, f_0 = f_0^\delta \in \mathcal{D}(\mathbb{F})\,, \qquad \text{for } k = 0, \cdots, k_* \\
&\qquad f_{k+1}^\delta = f_k^\delta + \gamma (F'_{c_k^\delta}(f_k^\delta))^* (g^\delta - F_{c_k^\delta}(f_k^\delta)) \\
&\qquad c_{k+1}^\delta \in \mathit{argmin}\ J_\alpha(c) := \left\| A_{f_{k+1}^\delta}(c) - g^\delta \right\|^2_{L^2(\Omega)} + \alpha \left\| c - c_k^\delta \right\|^2_{H^1(\Omega)}\,,
\end{aligned}
\tag{24}
$$

where, k_* is the stopping index, determine by the stopping criterion using the discrepancy principle

$$
\left\| \mathbb{F}(c_{k+1}^\delta, f_{k+1}^\delta) - g^\delta \right\|_{L^2(\Omega)} \le \tau\delta < \left\| \mathbb{F}(c_k^\delta, f_k^\delta) - g^\delta \right\|_{L^2(\Omega)}\,, \tag{25}
$$

and the relaxation parameter τ is such that

$$
\tau > 2\frac{1+\eta}{1-\eta}\,. \tag{26}
$$

Note that, if we have noise free data then $k_* = +\infty$. In this case we drop the index δ in (24).

Moreover, in (24) the positive parameter γ is a scaling factor to enforce convergence of the Landweber iteration [12, 4]. As a consequence of Lemma 8, we have

$$
f_{k+1}^\delta = f_k^\delta + \gamma V_k(x,T) = f_k^\delta + \gamma K^*(T)(g^\delta - u_k(x,T))\,,
$$

where $V_k(x,t)$ is the unique solution of (23) with $r_f = u_k(x,T) - g^\delta$ and $u_k(x,T) = u(c_k^\delta, f_k^\delta)$ is the unique solution of (1) with $c = c_k^\delta$ and $f = f_k^\delta$, respectively. Therefore, is enough that $\gamma < \|K(T)\|^{-2}$. Since the operator $-L$ generate a contraction semigroup, it follows that $\|K(T)\| \le T$. Therefore, is enough that $0 < \gamma < T^{-2}$. This estimate in not sharp.

It is worth noticing that in iterated Tikhonov approach, the parameter α do not play the rule of the regularization parameter [1]. In this case, we can choose any $\alpha > 6(\delta/\rho)^2$, where ρ is the radius (fixed) of the ball around c_0.

Remark 3. *In this remark we will discuss some point about the proposed iteration.*

- *The proposed algorithm (24) is a type of Kaczmarz strategy [11]. However, it is not the same Kaczmarz iteration proposed before in [7, 3, 1], since the unknown is a pair of parameter (c, f) and not a single parameter.*

- *Moreover, the proposed iteration is not the same as using a Landweber iteration [12] for the coordinate f_k and the iterated Tikhonov [1] for the coordinate c_k in the pair (c_k, f_k), since the iteration c_{k+1} depends of the iteration f_{k+1} as we can see in the iteration (24).*

- *We can mix some other type of iterative regularization methods in order to regularize the pair of parameters (c, f). However, the choice iterated Tikhonov in the second line in (24) imply that we can use the uniform discrepancy principle. The advantage of this choice is that, in each iteration, we only need to evaluate one time the residual, indeed, at the end of the cycle in the algorithm (24). See also (28) below. It saves significatively computational effort, compared with a discrepancy principle defined for each one of the lines of the system (24).*

5.1 Convergence Analysis

We start the analysis of the proposed algorithm with the following result that imply in the well posed of the iterative Tikhonov regularization.

Lemma 9. *For each $f \in D_f(\mathbb{F})$ fixed, there exists a minimizer of the Tiknonov functional J_α defined in (24).*

Proof. Note that $D_f(\mathbb{F})$ is convex and closed in $L^2(\Omega)$. Therefore it is weak sequentially closed. Now the proof follows immediately from the continuity of A_f given by Corollary 5. $\square$

Given the iteration formula in (24), we conclude that

$$c_{k+1}^\delta = c_k^\delta + \alpha^{-1}(A'_{f_{k+1}^\delta}(c_{k+1}^\delta))^*(g^\delta - A_{f_{k+1}^\delta}(c_{k+1}^\delta)) \,. \tag{27}$$

As usual for nonlinear Tikhonov type regularization, the global minimum for the Tikhonov functionals in (24) need not be unique. However, in [1] was proved that, for exact data, is possible to obtain convergence statements for any possible sequence of iterates, and we will accept any global solution. For noisy data, a (strong) semi-convergence result is obtained under the assumption that A_f has a Lipischitz Fréchet derivative as we have proved in Proposition 7.

Given the minimality of c_{k+1}^δ in the iteration (24), we have

$$\left\|\mathbb{F}(c_{k+1}^\delta, f_{k+1}^\delta) - g^\delta\right\|_{L^2(\Omega)} \le J_\alpha(c_{k+1}^\delta) \le J_\alpha(c_k^\delta) = \left\|F_{c_k^\delta}(f_{k+1}^\delta) - g^\delta\right\|_{L^2(\Omega)} . \tag{28}$$

Therefore, one important consequence of (28) is that, if the unified discrepancy principle (25) is not attained at the iteration $k+1$, then the standard discrepancy principle for Landweber iteration also is not attained, it is, while

$$\tau\delta \le \left\|\mathbb{F}(c_{k+1}^\delta, f_{k+1}^\delta) - g^\delta\right\|_{L^2(\Omega)} , \quad \text{then} \quad \tau\delta \le \left\|F_{c_k^\delta}(f_{k+1}^\delta) - g^\delta\right\|_{L^2(\Omega)} . \tag{29}$$

Because of this inequality, we call the discrepancy principle (25) a unified discrepancy principle.

Now, we are able to prove the convergence and stability of the iterative regularization method in (24).

Theorem 10. *Let $(c_0, f_0) = (c_0^\delta, f_0^\delta) \in \mathcal{D}(\mathbb{F})$ and the operators, $\mathbb{F}$, F_c and A_f as defined in (6) - (8) and τ as in (26). Then, for any $(c^*, f^*) \in \mathcal{D}(\mathbb{F})$ a solution of (6), the iteration given by (24) has the following properties:*

1. *While $\left\|g^\delta - \mathbb{F}(c_{k+1}^\delta, f_{k+1}^\delta)\right\|_{L^2(\Omega)} \ge \tau\delta$, we have that*

$$\begin{aligned} \left\|f^* - f_{k+1}^\delta\right\|_{L^2(\Omega)} &\le \left\|f^* - f_k^\delta\right\|_{L^2(\Omega)} \\ \left\|c^* - c_{k+1}^\delta\right\|_{L^2(\Omega)} &\le \left\|c^* - c_k^\delta\right\|_{L^2(\Omega)} . \end{aligned} \tag{30}$$

Moreover, if $(c_0, f_0) \in B_\rho(c^*, f^*) \subset \mathcal{D}(\mathbb{F})$, then $(c_k^\delta, f_k^\delta) \in B_{2\rho}(c^*, f^*)$ for all k and

$$\begin{aligned} k_*(\tau\delta)^2 \le \sum_{k=0}^{k_*-1} \left\|g^\delta - F_{c_k^\delta}(f_k^\delta)\right\|_{L^2(\Omega)}^2 &\le \frac{\tau\|f^* - f_0\|_{L^2(\Omega)}^2}{(1-2\eta)r - 2(1+\eta)} , \quad \forall\, 0 \le k \le k_* . \\ k_*(\tau\delta)^2 \le \sum_{k=0}^{k_*-1} \left\|g^\delta - A_{f_{k+1}^\delta}(c_{k+1}^\delta)\right\|_{L^2(\Omega)}^2 &\le \frac{\tau\|c^* - c_0\|_{L^2(\Omega)}^2}{(1-2\eta)r - 2(1+\eta)} , \quad \forall\, 0 \le k \le k_* . \end{aligned} \tag{31}$$

In particular, if $g^\delta = g$ (i.e., $\delta = 0$), then

$$\sum_{k=0}^{\infty} \|g - F_{c_k}(f_k)\|^2_{L^2(\Omega)} < \infty \qquad and \qquad \sum_{k=0}^{\infty} \left\|g - A_{f_{k+1}}(c_{k+1})\right\|^2_{L^2(\Omega)} < \infty\,. \tag{32}$$

2. *If there exist $(c^*, f^*) \in B_\rho((c_0, f_0))$, a solution of (6) and $\delta = 0$, then there exist a subsequence (c_k, f_k) given by (24) that converges to (c^*, f^*).*

3. *In the noisy data case, if the iterations are stopped according to the discrepancy principle (25) and τ is given by (26), then there exists a subsequence (that we denote by the same index) $(c^\delta_{k(\delta,g^\delta)}, f^\delta_{k(\delta,g^\delta)})$ that converges to a solution (c^*, f^*) of (6), as $\delta \to 0$.*

Proof. Let c^δ_k be fixed. Then, since $F_{c^\delta_k}$ is continuous, compact (see Corollary 5) and satisfies the local tangential cone condition (see Proposition 6), follows from [12, Chapter 2] that the sequence $f^\delta_k = f^\delta_k(c^\delta_k)$ given by the Landweber iteration satisfies the claim of the Theorem.

Now, let f^δ_k be fixed. Then, since $A_{f^\delta_{k+1}}$ is continuous, compact (see Corollary 5) and satisfies the local tangential cone condition (see Proposition 7), follows from [1] that the sequence $c^\delta_k = c^\delta_k(f^\delta_k)$ given by the iterative Tikhonov method satisfies the claim of the Theorem.

Therefore the convergence, stability and regularization properties of the approximated sequence (c^δ_k, f^δ_k) follows from a diagonal argument. □

6 Application in Thermography

Nowadays is well known that the body surface temperature is controlled by the blood perfusion, local metabolism and the heat exchange between the skin and the environment. Changes in any of these parameters can induce variations of temperature and heat fluxes at the skin surface. In particular, the apparition of malignant tumor imply in a highly vascularized skin region that lead an increase of local blood flow. Consequently, in a local increases of the blood perfusion and of the capacity of metabolic heat source [24, 23, 17].

Modern diagnostics of melanoma location in the skin are non invasive. They use the skin surface temperature measurements. However, this technique requires the solution of inverse bio-heat transfer problem. This problem consists in the simultaneous identification of thermal and geometrical parameters of tumor. In applications, some of the parameters that are interesting are the perfusion coefficient and the capacity of metabolic heat source [24, 19, 17, 23].

From the mathematical point of view the heat transfer processes in the domain of biological tissue are described by the Pennes [24, 20, 28, 21] equations

$$\rho C_p U_t - \nabla\cdot(a\nabla U) - \omega_b(x)\rho_b c_b(Q_0 - U) = Q_m(x)\,, \qquad \text{in } \Omega\times(0,T) \tag{33}$$

where ρ, C_p, a denotes density, specific heat, and thermal conductivity of tissue; ρ_b, c_b are density and specific heat of blood; ω_b blood perfusion rate; Q_m metabolic heat generation; Q_0 is the supplying arterial blood temperature and U the tissue temperature. Ω is the body region around the melanoma location. Therefore, $\Omega \subset \mathbb{R}^n$ for $n = 2$ or $n = 3$. For simplicity, we assume that the melanoma is located just below the skin and that we can consider $\Omega \subset \mathbb{R}^2$. For the case of $\Omega \subset \mathbb{R}^3$ the model is more complicated, principle, in terms of the boundary conditions [24, 20, 28, 21].

Besides the thermal parameters and metabolic rate of tissue, the skin temperature is also determined by many other factors such as the skin humidity, radiation emissivity of skin and parameters of surrounding air. These factors can be incorporated into the boundary condition at the skin surface. However, for simplicity, we assume that far from the tumor location, the heat effect of the tumor activity is insignificant. Therefore, the boundary conditions can be assumed to be constant and equal to Q_0. Moreover, without loss of generality, let us consider the parameters $\rho = C_p = \rho_b = c_b = 1$. Denoting $u = U - Q_0$ we have that u satisfies

$$\begin{aligned} u_t - \nabla \cdot (a \nabla u) + c(x)u = f(x)\,, \quad (x,t) \in \Omega \times (0,T) \\ u(x,t) = 0\,, \quad (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) = \varphi(x)\,, \end{aligned} \tag{34}$$

where $\varphi(x)$ imposes an initial spatial heating, $c(x) = \omega_b(x)$ and $f(x) = Q_m(x)$, that we assume be smooth as in the above sections. We assume that the temperature measurement, at final time, is given as in the equation (3).

Therefore, the theory developed before in this paper is applicable to the melanoma location in the body given the measurement on the skin surface in the following sense:

Theorem 11. *There exists a unique blood perfusion rate $\omega_b(x)$ and a unique metabolic heat generation $Q_m(x)$ for a given skin temperature measurement satisfying* (3).

Proof. Is a direct application of Theorem 3. □

Theorem 12. *Given measurement of the temperature on the skin surface satisfying* (9). *There exists an iterative regularization method that generates a sequence of approximate solutions for the identification of the blood perfusion rate $\omega_b(x)$ and the metabolic heat generation $Q_m(x)$ that is convergent and stable w.r.t. noise in the data.*

Proof. Consider the iterative regularization approach given by (24). Then, Theorem 10 imply the assertion. □

7 Conclusions and Further Works

In this work, we prove uniqueness of the spacewise parameter c and the source f in (1), for a given extra final time measurement (3). Moreover, we derive sufficient properties of the parameter-to-solution map to guarantee convergence and stability of approximated solutions obtained by the proposed iterative regularization method, if the stopping index is determined by the discrepancy principle (25). We also have analyzed the application of the theory developed here for a simplified version of the thermography model in melanoma identification. We are able to say that exists a unique blood perfusion rate and a unique metabolic heat generation in the simplified model (33) for a given final time measurement (3).

The next step in this line is the numerical implementation. In particular the analysis of the three-dimensional Pennes equation was not totally covered. In the three-dimensional case there are also many numerical difficulties, beyond the theoretical, that needs attention [24].

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A Appendix

In this appendix we will provide an auxiliary lemma that we need in the prove of Theorem 4.

Lemma A.1. *Let $\{\varphi_k\}_{k\in\mathbb{N}}$ be a sequence of functions, with $\underline{c} \leq \varphi_k(x) \leq \overline{c}$ for all $x \in \Omega$. If $\varphi_k \to \varphi$ in $L^p(\Omega)$ for some $p \in [1,\infty)$, then $\varphi_k \to \varphi$ in $L^p(\Omega)$ for all $p \in [1,\infty)$.*

Proof. We remark that, since Ω is bounded, $\varphi_k \in L^p(\Omega)$ for all $1 \leq p \leq \infty$.

Assume $\varphi_k \to \varphi$ in $L^2(\Omega)$.

Case 1.1 $(2 < p < \infty)$

For all $n, l \in \mathbb{N}$ with $n, l > k_0$

$$\int_\Omega |\varphi_n - \varphi_l|^p dx = \int_\Omega |\varphi_n - \varphi_l|^2 |\varphi_n - \varphi_l|^{p-2} dx \leq (2C)^{p-2} \int_\Omega |\varphi_n - \varphi_l|^2 dx\,.$$

Hence, $\{\varphi_k\}$ is a Cauchy sequence in $L^p(\Omega)$. Therefore, $\varphi_k \to \tilde{\varphi}$ in $L^p(\Omega)\,, 2 < p < \infty$.

Since, $L^p(\Omega)$ is continuously embedding in $L^2(\Omega)$ for $2 < p < \infty$, we have $\tilde{\varphi} \in L^2(\Omega) \cap L^p(\Omega)$ and

$$\|\varphi_k - \tilde{\varphi}\|_{L^2(\Omega)} \leq C \|\varphi_k - \tilde{\varphi}\|_{L^p(\Omega)}\,.$$

By the uniqueness of the limit $\varphi = \tilde{\varphi}$.

Case 1.2 $(1 \leq p \leq 2)$

For all $n, l \in \mathbb{N}$ with $n, l > k_0$

$$\int_\Omega |\varphi_n - \varphi_l|^p dx = \int_\Omega \left(|\varphi_n - \varphi_l|^2\right)^{\frac{p}{2}} dx \leq (meas(\Omega))^{p^*} \left(\int_\Omega |\varphi_n - \varphi_l|^2 dx\right)^{\frac{2}{p}}\,.$$

In other words, $\{\varphi_k\}$ is a Cauchy sequence in $L^p(\Omega)$ and hence, $\varphi^k \to \tilde{\varphi}$ in $L^p(\Omega)$ for $1 \leq p \leq 2$.

Claim. $\tilde{\varphi} \in L^2(\Omega) \cap L^p(\Omega)$.

$$\begin{aligned}\int_\Omega |\tilde{\varphi}|^2 dx &\leq C\left(\int_\Omega |\varphi_k - \tilde{\varphi}|^2 + |\varphi_k|^2 dx\right) \leq C\left(\int_\Omega \left(|\varphi_n - \varphi_l|^p\right)^{\frac{2}{p}} dx + \int_\Omega |\varphi_k|^2\right) dx \\ &\leq C\left(meas(\Omega))^{1-\frac{2}{p}} \int_\Omega |\varphi_k - \tilde{\varphi}|^p dx + C_1\right) < \infty\,.\end{aligned}$$

Hence, $\tilde{\varphi} \in L^2(\Omega)$. The continuous embedding of $L^2(\Omega)$ in $L^p(\Omega)$, for $1 \leq p < 2$, conclude the claim.

Now, given $\varepsilon > 0$, there exist $k \in \mathbb{N}$ large enough such that

$$\|\varphi - \tilde{\varphi}\|_{L^p(\Omega)} \leq \|\varphi_k - \varphi\|_{L^p(\Omega)} + \|\varphi_k - \tilde{\varphi}\|_{L^p(\Omega)} \leq C\|\varphi_k - \varphi\|_{L^2(\Omega)} + \|\varphi_k - \tilde{\varphi}\|_{L^p(\Omega)} < \varepsilon\,.$$

Therefore, $\varphi = \tilde{\varphi}$.

The arguments used in the proof of the reciprocal are similar to those presented above. Thus we will omit the proof. □

The next theorem is a version of Meyers's Theorem [18, Theorem 1] adapted to our case.

Theorem A.2. *[Meyers] Let Ω and the coefficients $(c, f) \in \mathcal{D}(\mathbb{F})$ as in Definition 1. Then, there exists a $p_0 > 2$ such that the unique solution $u = u(a, f)$ of (1) belongs to $L^2(0, T; W^{1,p}(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ for any $p \in [2, p_0[$.*

Proof. It follows from the classical parabolic partial differential theory that that $u \in L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; H_0^1(\Omega))$, [14]. Therefore, for each $t \in]0, T[$, $u(\cdot, t)$ satisfies the following elliptic equation

$$\begin{aligned} -\nabla(a(x)\nabla u(x,t)) + c(x)u(x,t)_t &= f(x)\,, x \in \Omega \\ u(x,t) &= 0 \text{ on } \partial\Omega\,. \end{aligned}$$

Follows from Meyers's Theorem [18, Theorem 1], that there exists a $p_0 > 2$ such that $u(\cdot, t) \in W^{1,p}(\Omega)$ for all $p \in [2, p_0[$. It proves the assertion. $\square$

References

[1] J. Baumeister, A. De Cezaro, and A. Leitao, *Modified iterated Tikhonov methods for solving systems of nonlinear ill-posed equations*, Inverse Probl. Imaging **5** (2011), no. 1, 1–17.

[2] A. De Cezaro and B. T. Johansson, *A note on uniqueness in the identification of a spacewise dependent source and diffusion coefficient for the heat equation*, submitted (2012), 1–17.

[3] A. De Cezaro, M. Hiltermiler, A. Leitao, and O. Scherzer, *On steepest-descent-Kaczmarz methods for regularizing systems of nonlinear ill-posed equtions*, Applied Mathematics and Computation **202** (2008), 596–607.

[4] H. W. Engl, M. Hanke, and A. Neubauer, *Regularization of inverse problems*, Mathematics and its Applications, vol. 375, Kluwer Academic Publishers Group, Dordrecht, 1996.

[5] N. L. Gol′dman, *Inverse problems with final overdetermination for parabolic equations with unknown coefficients multiplying the highest derivative*, Dokl. Akad. Nauk **438** (2011), no. 2, 162–167.

[6] ______, *On the properties of solutions of parabolic equations with unknown coefficients*, Differ. Uravn. **47** (2011), no. 1, 60–69.

[7] M. Haltmeier, A. Leitão, and O. Scherzer, *Kaczmarz methods for regularizing nonlinear ill-posed equations. I. Convergence analysis*, Inverse Probl. Imaging **1** (2007), no. 2, 289–298.

[8] V. Isakov, *Inverse problems for partial differential equations*, second ed., Applied Mathematical Sciences, vol. 127, Springer, New York, 2006.

[9] M. Ivanchov, *Inverse problems for equations of parabolic type*, Mathematical Studies Monograph Series, vol. 10, VNTL Publishers, L′viv, 2003.

[10] B. T. Johansson and D. Lesnic, *A procedure for determining a spacewise dependent heat source and the initial temperature*, Appl. Anal. **87** (2008), no. 3, 265–276.

[11] S. Kaczmarz, *Approximate solution of systems of linear equations*, Internat. J. Control **57** (1993), no. 6, 1269–1271, Translated from the German.

[12] B. Kaltenbacher, A. Neubauer, and O. Scherzer, *Iterative regularization methods for nonlinear ill-posed problems*, Radon Series on Computational and Applied Mathematics, vol. 6, Walter de Gruyter GmbH & Co. KG, Berlin, 2008.

[13] M. V. Klibanov and A. Timonov, *Carleman estimates for coefficient inverse problems and numerical applications*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 2004.

[14] O. A. Landyzenskaya, V. A. Solonikov, and N. N. Urealceva, *Linear and quasilinear equations of parabolic type*, Translations of Mathematical Monographs vol 23, AMS, Providence, RI, 1968.

[15] M. M. Lavrentiev, A. V. Avdeev, M. M. Lavrentiev, Jr., and V. I. Priimenko, *Inverse problems of mathematical physics*, Inverse and Ill-posed Problems Series, VSP, Utrecht, 2003.

[16] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York, 1972, Translated from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.

[17] Xu L.X. Liu, J., *Boundary information based diagnostics on the thermal states of biological bodies*, International Journal of Heat and Mass Transfer (2000), no. 43, 28272839.

[18] N. G. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa (3) **17** (1963), 189–206.

[19] M. Paruch and E. Majchrzak, *Identification of tumor region parameters using evolutionary algorithm and multiple reciprocity boundary element method*, Engineering Applications of Artificial Intelligence (2007), no. 20, 647–655.

[20] H.H. Pennes, *Analysis of tissue and arterial temperatures in the resting human forearm*, Journal of Applied Physiology (1948), no. 1, 93–122.

[21] R. Romero-Méndez, J. N. Jiménez-Lozano, M. Sen, and F. J. González, *Analytical solution of the Pennes equation for burn-depth determination from infrared thermographs*, Math. Med. Biol. **27** (2010), no. 1, 21–38.

[22] W. Rundell, *Determination of an unknown nonhomogeneous term in a linear partial differential equation from overspecified boundary data*, Applicable Anal. **10** (1980), no. 3, 231–242.

[23] C.W. Song, A. Lokshina, J.G. Rhee, M. Patten, and S.H. Levitt, *Implication of blood fow in hyperthermic treatment of tumors*, IEEE Trans. Bio-Med. Eng. (1984), no. 31, 9–16.

[24] D. Trucu, D. B. Ingham, and D. Lesnic, *Space-dependent perfusion coefficient identification in the transient bio-heat equation*, J. Engrg. Math. **67** (2010), no. 4, 307–315.

[25] S. Vessella, *Quantitative estimates of unique continuation for parabolic equations, determination of unknown time-varying boundaries and optimal stability estimates*, Inverse Problems **24** (2008), no. 2, 023001, 81.

[26] M. Yamamoto, *Carleman estimates for parabolic equations and applications*, Inverse Problems **25** (2009), no. 12, 123013, 75.

[27] K. Yosida, *Functional analysis*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the sixth (1980) edition.

[28] J. J. Zhao, J. Zhang, N. Kang, and F. Yang, *A two level finite difference scheme for one dimensional Pennes' bioheat equation*, Appl. Math. Comput. **171** (2005), no. 1, 320–331.