A note on uniqueness in the identification of a spacewise dependent source and diffusion coefficient for the heat equation

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We investigate uniqueness in the inverse problem of reconstructing simultaneously a spacewise conductivity function and a heat source in the parabolic heat equation from the usual conditions of the direct problem and additional information from a supplementary temperature measurement at a given single instant of time. In the multi-dimensional case, we use Carleman estimates for parabolic equations to obtain a uniqueness result. The given data and the solution domain are sufficiently smooth such that the required norms and derivatives of the conductivity, source and solution of the parabolic heat equation exist and are continuous throughout the solution domain. These assumptions can be further relaxed using more involved estimates and techniques but these lengthy details are not included. Instead, in the special case of the one-dimensional heat equation, we give an alternative and rather straightforward proof of uniqueness for the inverse problem, based on integral representations of the solution together with density results for solutions of the corresponding adjoint problem. In this case, the required regularity conditions on the conductivity, source and the solution of the parabolic heat equation are weakened to classes of integrable functions.

Keywords: uniqueness; spacewise conductivity and source; final time measurements; heat equation; Carleman estimates.

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1 Introduction

Inverse problems of parameter identification in partial differential equations have several important applications including thermal prospection of material and bodies, hydraulic prospecting of soil, photonic detection of cancer, finding pollution sources, see, for example, [4, 7, 9, 10, 11, 18, 20, 21] and references therein. For inverse problems in general it is important to find and specify appropriate data such that the set of parameters to be reconstructed are uniquely identifiable. We shall consider an inverse problem for the parabolic heat equation, where the additional data is information about the solution obtained from a spacewise measurement at the final time. To be more specific, we shall

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show uniqueness of the identifiability of a pair of functions $(a(x), f(x))$, representing the spacewise thermal conductivity and the heat source, in the parabolic heat equation

$$\begin{aligned} u_t - \nabla \cdot (a(x)\nabla u) &= f(x) \text{ in } \Omega \times (0,T) \\ u(x,t) &= 0 \text{ for } (x,t) \in \partial\Omega \times (0,T) \\ u(x,0) &= h(x) \text{ for } x \in \Omega, \end{aligned} \tag{1}$$

for a given initial temperature h, with the additional temperature measurement g at time $t = T$, i.e.

$$u(x,T) = g(x) \text{ for } x \in \Omega\,, \quad T > 0\,. \tag{2}$$

We assume that a and f are spacewise dependent real-valued functions and that there exists $\underline{a} > 0$ such that $a(x) \geq \underline{a}$ for all $x \in \Omega$. This implies that the operator $Lu = -\nabla \cdot (a(x)\nabla u)$ is elliptic. For the moment we assume that the coefficient, the initial and final conditions are sufficiently smooth such that there exist a unique classical solution $u(x,t)$ of the problem (1) and that the required compatibility conditions are satisfied. For the precise statement of smoothness and other assumptions to guarantee the existence of such a classical solution of the direct problem (1), see [15, Theorem 5.2]. Regularity conditions will be further discussed in Section 2 and in the Appendix A.

There are many contributions on uniqueness for the identifiability of coefficients in parabolic type equations in the case of lateral overdetermination. Since the literature on this subject is vast we suggest the reader to consult [9, 10, 13, 18, 21] and references therein.

On the other hand, there are only a few papers related to the inverse problem of spacewise coefficient identification with given additional measurements at the final time. Indeed, uniqueness from final time data for a spacewise dependent heat source was shown in [19] and simultaneous determination of a heat source and initial data from spacewise measurements was investigated in [12]. Uniqueness from final time data for the identification of diffusion coefficients was shown in a recent paper [5], for the one-dimensional heat equation. We point out that the arguments proving uniqueness for the one-dimensional setting [5] appear not possible to extend to the multi-dimensional case.

In [21], Carleman type estimates were used to prove uniqueness and stability of a sufficiently smooth diffusion coefficient in the heat equation, with measurements given at an intermediate time. However, to the authors' knowledge there are no results on uniqueness for the reconstruction of both a spacewise dependent conductivity and heat source. Thus, we shall state and prove such a uniqueness result building on Carleman estimates in [21]. For the ease of presentation and to highlight the usefulness of Carleman estimates for the inverse problem (1)–(2), we shall simplify the proof using smoothness assumptions together with some a priori knowledge of the conductivity close to the boundary of the solution domain. It is conjectured that these smoothness assumptions can be removed but we only indicate possible generalizations rather than give full lengthy and complex details. To convince the reader that it is possible to have uniqueness also in spaces of integrable functions, we give a rather straightforward proof of uniqueness in the case of a one-dimensional solution domain, where the smoothness assumptions are relaxed and more general. This proof is based on a recent work [5, Section 5] and involves integral relations obtained from Green's formula for the solution together with some denseness properties of the adjoint (heat) equation. This technique does not appear possible though to generalize to higher dimensions.

We point out that once uniqueness is shown for smooth data, one can use standard approximation techniques together with stability results for parabolic equations to get a result for also for non-smooth data.

Note that it is crucial to have only spacewise dependence in the heat conductivity and source term; there are examples showing non-uniqueness of the reconstruction of the heat source in the case of time-dependent sources, see [2, 6, 8]. A simple counterexample to the uniqueness in the case of a time-dependent heat conductivity is presented in [5, Section 5].

We point out that an interesting related problem that we do not explore further in this contribution is conditional stability estimates for the above inverse problem (1)–(2). In general, for an inverse problem, in spite of the ill-posedness conditional stability estimates assure that one can restore the stability of the requested (physical) parameters provided they are restricted to some class within an a priori bounded set. The conditional stability is not only of theoretical interest but is also of importance for the construction of numerically stable solutions. For example, in [3] a stability estimate gives convergence rates for certain Tikhonov regularized solutions. There are several methods in the literature for proving conditional stability, see [9, 21]. The method of Carleman estimates is one possibility for obtaining conditional stability. Thus, it is possible to obtain such stability based on the results presented in the present paper. We do not go into the details here, it is deferred to future work. The reader can further consult [9, 18, 21] and the references therein.

For the outline of this paper, we show a uniqueness result for the above inverse problem using Carleman estimates, see Section 2 and Theorem 4. In Section 3, we address global uniqueness for the identification of the heat source and conductivity in the one-dimensional case, using density arguments for solutions of the corresponding adjoint problem. The arguments in Section 3 weakens the smoothness assumptions of the spacewise source and coefficients in the inverse problem (1)–(2). In the final section, we draw some conclusions and discuss some possible generalization of the presented uniqueness results. For the sake of completeness, in the Appendix A, we prove that for a given class of initial data, the solution of the heat equation at the final time satisfies the required assumptions stated in Section 2.

Notation: We finish the introduction stating some notation that we use: The set Ω is an open and bounded subset of $\mathbb{R}^n$, with the boundary $\partial\Omega$ being at least Lipschitz smooth. By $L^p(\Omega)$ for $1 \leq p < \infty$, we denote the usual space of p-integrable functions on Ω with the usual norm $\|\cdot\|_{L^p(\Omega)}$. The space $L^\infty(\Omega)$ is the standard L^∞-space. We denote by $W^{k,p}(\Omega)$ the standard Sobolev space on Ω with generalized derivatives of order $\leq k$ in $L^p(\Omega)$. In particular, for $p = 2$ we have the Hilbert spaces $H^k(\Omega)$. Moreover, since $\partial\Omega$ is Lipschitz, the trace of a function in $H^1(\Omega)$ to the boundary is well-defined.

Let $T > 0$ be fixed and define the measurable function $u(\cdot, t) : (0, T) \longrightarrow X$, where X is a Banach space. We denote by $C([0, T]; X)$ the space of continuous mappings $u(\cdot, t)$ with the usual norm and by $L^2((0, T); X)$ the space of functions such that

$$\int_0^T \|u(\cdot, t)\|_X^2 \, dt < \infty \, .$$

We also assume that a and f are sufficiently regular spacewise real-valued functions with $0 < \underline{a} \leq a(x)$ for every $x \in \Omega$. Therefore, the differential operator

$$Lu = \nabla(a(x)\nabla u)$$

is elliptic. Since, the operator $-L$ generate a contraction semigroup, there exists a unique solution u of (1) with

$$\|u\|_1 = \int_0^T \left(\|u_t(\cdot,t)\|^2_{L^2(\Omega)} + \|u(\cdot,t)\|^2_{H^2(\Omega)}\right) dt < \infty\,.$$

2 Local uniqueness for the spacewise source and heat conductivity in (1)–(2)

As mentioned, we do not strive to obtain the most general result, but we only wish to convince the reader that there can be at most one spacewise dependent heat conductivity coefficient and one spacewise dependent heat source that together satisfy the given final time data, i.e. that solve the inverse problem (1)–(2). Therefore, we assume that the given data and the solution domain are sufficiently smooth such that the required norms and derivatives of the coefficients, sources and solution of (1)–(2) exist and are continuous throughout the solution domain. For the precise statement of smoothness and compatibility conditions for the parabolic heat equation, see [15, Theorem 5.2]. Once uniqueness is shown for smooth data, one can use standard approximation techniques together with stability results for parabolic equations to get a result for non-smooth data. Note that there are now many solvability results and estimates for parabolic equations with very general coefficients, see further [15, 14].

For our proof of uniqueness we use local Carleman estimates and follow [21, Section 6]. To simplify the presentation further and to avoid cut-off functions and global Carleman estimates, we assume that the diffusion coefficient is known in a region $\Omega - \overline{D}$, where $\overline{D} \subset\subset \Omega$, and $0 < |D| < \infty$. In other words,

Assumption 1. *The diffusion coefficient $a(x)$ in (1) is known for every $x \in \Omega \setminus \overline{D}$, where $D \subset \Omega$ with ∂D sufficiently smooth and $d(D, \partial\Omega) > \gamma > 0$.*

This is a reasonable assumption in applications, since the material (body) Ω might be coated or layered and the physical properties of the outer layer is known. The above assumption forces any two solutions of the inverse problem (1)–(2) to be equal in $\Omega \setminus \overline{D}$, i.e. the difference has compact support in Ω. More precisely, we have the following lemma:

Lemma 2. *Let $u = u(a, f_1)$ and $v = v(b, f_2)$ be solutions of (1)–(2), with spacewise heat conductivities a and b and spacewise heat sources f_1 and f_2, respectively. Assume that $a = b$ in $\Omega - \overline{D}$. Then the difference $w = u - v$ is such that $w = 0$ in $\Omega \setminus \overline{D}$. Moreover $f_1 = f_2$ in $\Omega - \overline{D}$.*

Proof. By linearity of (1), w satisfies

$$\begin{aligned} w_t - \nabla \cdot (a(x)\nabla w) &= \nabla \cdot ((a-b)\nabla v) + (f_1 - f_2)(x) \text{ in } \ \Omega \times (0,T) \\ w(x,t) &= 0 \text{ for } (x,t) \in \partial\Omega \times (0,T) \\ w(x,0) &= w(x,T) = 0 \text{ for } x \in \Omega. \end{aligned} \tag{3}$$

From the assumption that $a = b$ in $\Omega - \overline{D}$ we have,

$$\begin{aligned} w_t - \nabla \cdot (a(x)\nabla w) &= (f_1 - f_2)(x) \text{ in } \Omega - \overline{D} \times (0,T) \\ w(x,t) &= 0 \text{ for } (x,t) \in \partial\Omega \times (0,T) \\ w(x,t) &= g_0(x) \text{ for } (x,t) \in \partial D \times (0,T) \\ w(x,0) &= w(x,T) = 0 \text{ for } x \in \Omega. \end{aligned} \tag{4}$$

Therefore, using standard parabolic theory we conclude that there exist only one $w \in C([0,T], H_0^1(\Omega) \times H^2(\Omega)$ solution of (4).

Let we define the following problems

$$\begin{aligned} w_t^{(1)} - \nabla \cdot (a(x)\nabla w^{(1)}) &= (f_1 - f_2)(x) \text{ in } \Omega - \overline{D} \times (0,T) \\ w^{(1)}(x,t) &= 0 \text{ for } (x,t) \in \partial\Omega \times (0,T) \\ w^{(1)}(x,t) &= 0 \text{ for } (x,t) \in \partial D \times (0,T) \\ w^{(1)}(x,0) &= w^{(1)}(x,T) = 0 \text{ for } x \in \Omega, \end{aligned} \tag{5}$$

Since the source is in $L^2(\Omega)$, we have from [19, Theorem 2] that there exist a unique solution $w^{(1)}$ of (5).

Therefore, taking the difference $w^{(2)} = w - w^{(1)}$, we find that $w^{(2)}$ exists and satisfies

$$\begin{aligned} w_t^{(2)} - \nabla \cdot (a(x)\nabla w^{(2)}) &= 0 \text{ in } \Omega - \overline{D} \times (0,T) \\ w^{(2)}(x,t) &= 0 \text{ for } (x,t) \in \partial\Omega \times (0,T) \\ w^{(2)}(x,t) &= g_0(x,t) \text{ for } (x,t) \in \partial D \times (0,T) \\ w^{(2)}(x,0) &= w^{(2)}(x,T) = 0 \text{ for } x \in \Omega. \end{aligned} \tag{6}$$

We start by considering the solution $w^{(1)}$ of (5). From [19, Theorem 1] given a final condition at time $T > 0$ there is at most one spacewise dependent source giving rise to this final time value for the heat equation with homogeneous boundary and initial condition. Thus, since $w^{(1)}$ is zero at $t = T$ and has homogeneous boundary and initial condition we conclude that $f_1 - f_2 = 0$, this in turn implies that $w^{(1)} = 0$ in $\Omega - \bar{D} \times (0,T)$.

We then consider the solution $w^{(2)}$ of (6). If the boundary data g_0 is identically zero it is clear that $w^{(2)} = 0$. Therefore, assume that g_0 is not identically zero. Without loss of generality, we can assume that $g_0(x,t) > 0$ at a point (x,t) of the boundary $\partial D \times (0,T)$ and since g_0 is at least continuous it follows that $g_0(x,t) > 0$ in $E \times (t_1,t_2)$, with E being a surface patch of ∂D. This in turn implies that $w^{(2)}(x,t) > 0$ for (x,t) sufficiently close to this open set. Applying [7, Theorem 9.2] it follows that $w^{(2)}(x,s) > 0$ for every $t < s \leq T$ contradicting the given final condition $w^{(2)}(x,T) = 0$. Therefore, $g_0 = 0$ implying that $w^{(2)} = 0$.

Thus, since $w = w^{(1)} + w^{(2)}$ we have $w = 0$ and therefore the assertion of the lemma is proved. □

Note that, if the measurements are given at an intermediate time $0 < t_0 < T$, the conclusions of Lemma 2 may not be true.

2.1 Some additional assumptions and admissible solutions

In order to proceed to the next step in the proof of uniqueness of the spacewise dependent pair (a, f) satisfying (1)–(2) we need some additional assumptions. We assume that the admissible set of unknown elements is

$$\mathcal{A} := \{(a, f) \in C^2(\overline{\Omega}) \, : \, a(x) > \underline{a} > 0 \, , x \in \Omega \, , \quad \|a\|_{C^2(\overline{\Omega})} + \|f\|_{C^2(\overline{\Omega})} \leq M\} \, , \tag{7}$$

where $M > 0$ is an arbitrary fixed constant. Moreover, we assume that g and h are smooth enough such that we can take the t-derivatives of $w = u - v$ for the time points needed, see [15] for the details on the required smoothness assumptions on g and h for this to be the case. Note that in Section 3 and in Appendix A we will provide more details on the regularity of the parameters and the input data. Moreover, we also assume that $\partial\Omega$ is smooth enough such that $\partial\Omega \subset \{x_1 = 0\}$. In fact, the boundary can be covered by a finite number of surface patches mapping to this region and it is therefore enough to consider the estimates in this half-space. The general results can be obtained in standard way for partial differential equations by using a partition of unity argument.

We remark that since the coefficients and the source term are time-independent, the solution of (1) is analytic in time [9, 20]. Moreover, the compact support of w in Ω guaranteed by Lemma 2 implies the follows boundary conditions

$$\begin{aligned} w_t(x,t) &= 0 \quad (x,t) \in \partial\Omega \times (0,T) \, , \\ \frac{\partial}{\partial \eta} w_t(x,t) &= 0 \quad (x,t) \in \partial\Omega \times (0,T) \, , \\ \frac{\partial}{\partial \tau} w_t(x,t) &= 0 \quad (x,t) \in \partial\Omega \times (0,T) \, , \end{aligned} \tag{8}$$

where η is the outer normal vector and τ is the tangential normal vector at $\partial\Omega$. Moreover, $\frac{\partial}{\partial\eta} w_t(x,t) = \eta(x) \cdot \nabla w_t(x, \cdot)$ and $\frac{\partial}{\partial\tau} w_t(x,t) = \tau(x) \cdot \nabla w_t(x, \cdot)$ are the normal and tangential derivatives of w_t for $x \in \partial\Omega$.

We shall present a uniqueness result for the inverse problem (1)–(2) based on local Carleman estimates. A crucial step for such estimates is the construction of suitable weight functions, $\varphi(x,t)$ and $\beta(x,t)$. For a given domain D we can apply the arguments in [21, Section 5.1] or the arguments in [18, Section 2], with a special choice of a function $d(x)$ such that

$$\varphi(x,t) = e^{\lambda\beta(x,t)} \tag{9}$$

and $\beta(x,t) = d(x) + e(t)$, where $\max_{t\in[0,T]} \varphi(x,t) = \varphi(x,T)$. Moreover, we can construct φ and β with

$$Q := \{(x,t) \, : \, x_1 > 0 \, , \, \varphi(x,t) > e^{-\lambda\delta}\} = \{(x,t) \, : \, x_1 > 0 \, , \, \beta(x,t) > -\delta\} \, , \tag{10}$$

and $Q \cap \{t = T\} = D$. In (9) and (10), $\lambda, \delta > 0$ are some fixed constants being sufficiently large.

Let we give a very simple example for choice $d(x)$ and $e(t)$ form [21, Section 6].

Example 3. *Set $x' = (x_2, \cdot, x_n)$ and $x = (x_1, x') \in \mathbb{R}^n$. Let we assume that $D = D(\delta) := \{(x, x'); 0 < x_1 < -|x'|^2/\gamma + \delta/\gamma\}$ and moreover $D(4\delta) \subset \Omega$, for some $\gamma > 0$ and $\delta > 0$. Define $d(x) := -\gamma x_1 - |x'|^2$ and $e(t) := -(t-T)^2$. Therefore, we have by (9) that $\max_{t\in[0,T]} \varphi(x,t) = \varphi(x,T)$ and that $Q = Q(\delta)$ satisfies $Q \cap \{t = T\} = D$.*

As mentioned earlier, since we have spacewise dependent coefficients, the solution of (1) is analytic in time. Therefore, the solution can be extended beyond the final time T, see [20, Section 3] and T can therefore be considered as an interior point, as is required in the derivation of the results in [21, Section 6].

We remark that one can try to identify the most general set of conductivity coefficients having minimal regularity assumptions together with minimal regularity of the boundary of Ω for which the weight functions φ and β do exist. However, as pointed out earlier this is not the main aim of this study, we shall only present a proof in the case of smooth and regular solutions and domains. For a discussion about more general function spaces and domains for which this derivation can hold true, see [18, 21] and references therein.

We also remark that since w has compact support (see Lemma 2), we do not need to use cut-off functions. Therefore, many of the calculations and terms in [21, Theorem 6.1] can be dropped and the steps in the proof become easier to follow. However, since we have two unknowns, some challenges and adjustments do remain, and we present the steps below.

2.2 A uniqueness proof

Let $u = u(a, f_1)$ and $v = v(b, f_2)$ be solutions of (1)–(2), with spacewise heat diffusions a and b and spacewise heat sources f_1 and f_2, respectively, and put $w = u - v$. Denote by $z := w_t$; a well-defined quantity due to the smoothness assumptions above and since we only work with classical solutions. Given the analyticity of w in time and the assumption that the unknown pair (a, f) is time independent, it follows that z satisfies

$$z_t - \nabla \cdot (a(x)\nabla z) = \nabla \cdot ((a - b)\nabla v_t) \tag{11}$$

with homogeneous boundary, initial and final conditions. Then, [21, Theorem 3.2] implies

$$\int_Q \left(\frac{1}{s} \sum_{i,j=1}^{n} |\partial_i \partial_j z|^2 + s|\nabla z|^2 + s^2|z|^2 \right) e^{2s\varphi}\, dxdt \le C \int_Q |\nabla \cdot ((a - b)\nabla v_t)|^2 e^{2s\varphi}\, dxdt \tag{12}$$

for all sufficiently large $s > 0$. Note that the integral over the boundary in [21, Theorem 3.2] is identically zero due to (8).

Now we shall derive an estimate of the right-hand side of (12). A direct calculation of $\nabla \cdot ((a - b)\nabla v_t)$ implies that

$$|\nabla \cdot ((a - b)\nabla v_t)|^2 \le (|\nabla(a - b)|^2 + |a - b|^2)(|\nabla v_t|^2 + |\triangle v_t|^2)\,. \tag{13}$$

Hence,

$$\int_Q |\nabla \cdot ((a - b)\nabla v_t)|^2 e^{2s\varphi}\, dxdt \le C(||\nabla v_t||_\infty^2 + ||\triangle v_t||_\infty^2) \int_Q (|\nabla(a - b)|^2 + |a - b|^2) e^{2s\varphi}\, dxdt\,. \tag{14}$$

Since u, v are assumed smooth enough (see [15] for necessary requirement on the data) we have that $(||\nabla v_t||_\infty^2 + ||\triangle v_t||_\infty^2) < \infty$. Therefore, $C = (||\nabla v_t||_\infty^2 + ||\triangle v_t||_\infty^2)$ is finite.

Combining (12) and (14) imply

$$\int_Q \left(\sum_{i,j=1}^{n} |\partial_i \partial_j z|^2 + s^2|\nabla z|^2 + s^3|z|^2 \right) e^{2s\varphi}\, dxdt \le C \int_Q s(|\nabla(a - b)|^2 + |a - b|^2) e^{2s\varphi}\, dxdt \tag{15}$$

for all sufficiently large $s > 0$.

Given the final time measurement (2), equation (3) gives, at $t = T$,

$$\begin{aligned} \nabla \cdot ((a-b)\nabla g(x)) &= w_t(x,T) + (f_1 - f_2)(x) \\ \nabla(\nabla \cdot ((a-b)\nabla g(x))) &= \nabla w_t(x,T) + \nabla (f_1 - f_2)(x)\,. \end{aligned} \tag{16}$$

It follows from a straightforward manipulation of (16) together with integration that

$$\begin{aligned} &\int_D (|f_1 - f_2|^2 + |\nabla(f_1 - f_2)|^2) e^{2s\varphi(x,T)}\,dx \\ &\qquad + \int_D (|\nabla \cdot ((a-b))\nabla g(x)|^2 + |\nabla(\nabla \cdot ((a-b)\nabla g(x)))|^2) e^{2s\varphi(x,T)}\,dx \\ &\qquad \le \int_D (|w_t(x,T)|^2 + |\nabla w_t(x,T)|^2) e^{2s\varphi(x,T)}\,dx \\ &\qquad + 2\int_D |(\nabla \cdot ((a-b)\nabla g(x))(f_2 - f_1)(x))| e^{2s\varphi(x,T)}\,dx \\ &\qquad + 2\int_D |(\nabla(\nabla \cdot ((a-b)\nabla g(x)))\nabla(f_2 - f_1)(x))| e^{2s\varphi(x,T)}\,dx. \end{aligned} \tag{17}$$

We put

$$A = 2\int_D |(\nabla \cdot ((a-b)\nabla g(x))(f_2 - f_1)(x))| e^{2s\varphi(x,T)}\,dx$$

and

$$B = 2\int_D |(\nabla(\nabla \cdot ((a-b)\nabla g(x)))\nabla(f_2 - f_1)(x))| e^{2s\varphi(x,T)}\,dx\,.$$

Note that

$$\|\partial_j w_t e^{s\varphi}\|^2_{H^1(Q)} \le \int_Q \left(\sum_{i,j=1}^n |\partial_i \partial_j w_t|^2 + s^2 |\nabla w_t|^2 + s^3 |w_t|^2 \right) e^{2s\varphi}\,dxdt. \tag{18}$$

Applying the trace theorem [1] in Q and noting that $Q \cap \{t = T\} = D$ we have

$$\int_D (|w_t(x,T)|^2 + |\nabla w_t(x,T)|^2) e^{2s\varphi(x,T)}\,dx \le C \sum_{j=0}^{1} \|\partial_j w_t e^{s\varphi}\|^2_{H^1(Q)}\,. \tag{19}$$

Now, (15), (17), (18) and (19) yield

$$\begin{aligned} &\int_D (|f_1 - f_2|^2 + |\nabla(f_1 - f_2)|^2) e^{2s\varphi(x,T)}\,dx \\ &\qquad + \int_D (|\nabla \cdot ((a-b))\nabla g(x)|^2 + |\nabla(\nabla \cdot ((a-b)\nabla g(x)))|^2) e^{2s\varphi(x,T)}\,dx \\ &\qquad \le C \int_Q s(|\nabla(a-b)|^2 + |a-b|^2) e^{2s\varphi(x,t)}\,dxdt + (A+B)\,. \end{aligned} \tag{20}$$

The next step in the proof is an estimate of the second integral in the left hand-side of equation (20).

Lemma 4. *[21, Lemma 6.1] Let $(a-b) \in H^2(\overline{\Omega})$ such that $|a-b| = |\nabla(a-b)| = 0$ on $\partial\Omega$. Assume that*

$$\gamma\partial_1 g(x) + 2\sum_{j=2}^{n}(\partial_j g)(x) \le 0 \quad x \in \overline{\Omega}\,, \tag{21}$$

$$\partial_1 g(x) > 0\,, \quad x \in \partial\Omega.$$

Then there exists a constant $C > 0$ such that

$$\int_\Omega s^2(|\nabla(a-b)|^2 + |a-b|^2)e^{2s\varphi(x,T)}\,dx \le C\int_\Omega(|\nabla\cdot((a-b))\nabla g(x)|^2 + |\nabla(\nabla\cdot((a-b)\nabla g(x)))|^2)e^{2s\varphi(x,T)}\,dx \tag{22}$$

for all sufficiently large $s > 0$.

Note that, since we assume that $a = b$ in $\Omega - \overline{D}$, the estimate (22) holds over D. This in turn using equation (20) and Lemma 4 give

$$\int_D(|f_1 - f_2|^2 + |\nabla(f_1 - f_2)|^2)e^{2s\varphi(x,T)}\,dx + \int_D s^2(|\nabla(a-b)|^2 + |a-b|^2)e^{2s\varphi(x,T)}\,dx \le C\int_Q s(|\nabla(a-b)|^2 + |a-b|^2)e^{2s\varphi(x,t)}\,dxdt + (A+B) \tag{23}$$

for all sufficiently large $s > 0$. Since $\varphi(x,t) \le \varphi(x,T)\,, x \in (\overline{\Omega})\,, \quad 0 \le t \le T$, we have

$$\int_D(|f_1 - f_2|^2 + |\nabla(f_1 - f_2)|^2)e^{2s\varphi(x,T)}\,dx + \int_D s^2(|\nabla(a-b)|^2 + |a-b|^2)e^{2s\varphi(x,T)}\,dx \le CT\int_D s(|\nabla(a-b)|^2 + |a-b|^2)e^{2s\varphi(x,T)}\,dxdt + (A+B) \tag{24}$$

for all sufficiently large $s > 0$.

The final step in the proof of uniqueness is to obtain an estimate of the quantity $A + B$ in (24) in terms of the coefficients of the first equation in (3). We assume that

$$\|\nabla(\triangle g(x))\|_\infty + \|\triangle g(x)\|_\infty + \|\nabla g\|_\infty < \infty\,. \tag{25}$$

Using the Young inequality with $\varepsilon > 0$ we have

$$A = 2\int_D |(\nabla\cdot((a-b)\nabla g(x))(f_2 - f_1)(x))|e^{2s\varphi(x,T)}\,dx \le \frac{2}{\varepsilon}\int_D |(\nabla\cdot((a-b)\nabla g(x))|^2 e^{2s\varphi(x,T)}\,dx + \varepsilon\int_D |(f_2 - f_1)(x))|^2 e^{2s\varphi(x,T)}\,dx$$

and

$$B = 2\int_D |(\nabla(\nabla\cdot((a-b)\nabla g(x)))\nabla(f_2 - f_1)(x))|e^{2s\varphi(x,T)}\,dx \le \frac{2}{\varepsilon}\int_D |(\nabla(\nabla\cdot((a-b)\nabla g(x)))|^2 e^{2s\varphi(x,T)}\,dx + \varepsilon\int_D |\nabla(f_2 - f_1)(x))|^2 e^{2s\varphi(x,T)}\,dx.$$

Using the similar estimate as before for $|(\nabla\cdot((a-b)\nabla g(x))|$, see (14), in combination with assumption (25) we have

$$A < \frac{C}{\varepsilon}\int_D (|\nabla(a-b)|^2 + |a-b|^2)e^{2s\varphi(x,T)}\,dx + \varepsilon\int_D |(f_2-f_1)(x))|^2 e^{2s\varphi(x,T)}\,dx.$$

Similarly, a direct calculation of $(\nabla(\nabla\cdot((a-b)\nabla g(x)))$ using the assumption (25) imply

$$\int_D |(\nabla(\nabla\cdot((a-b)\nabla g(x)))|^2 e^{2s\varphi(x,T)}\,dx \le C\int_D (|\triangle(a-b)|^2 + |\nabla(a-b)|^2 + |a-b|^2)e^{2s\varphi(x,T)}\,dx.$$

We then choose $0<\varepsilon<1/2$. Using the above estimates for A and B in (24) and put together this give

$$\begin{aligned}\int_D (|f_1-f_2|^2 + |\nabla(f_1-f_2)|^2)e^{2s\varphi(x,T)}\,dx + \int_D s^2(|\nabla(a-b)|^2+|a-b|^2)e^{2s\varphi(x,T)}\,dx \\ \le CT\int_D [(s+1)(|\nabla(a-b)|^2+|a-b|^2) + |\triangle(a-b)|^2]e^{2s\varphi(x,T)}\,dxdt \qquad (26)\end{aligned}$$

for all sufficiently large $s>0$.

For $s>0$ sufficiently large the terms in the right-hand side can be absorbed by the corresponding terms in left hand-side. Indeed, by assumption, $|\triangle(a-b)|^2 \le M^2$ and therefore it is enough to choose $s>0$, such that $s^2(|\nabla(a-b)|^2+|a-b|^2) > CT(s+1)(|\nabla(a-b)|^2+|a-b|^2)+M^2$. Thus, with such a choice of $s>0$, we conclude that the second integral in the left-hand side is identically zero, i.e. $a=b$ also in D. The inequality (26) then implies that also the sources are equal, i.e. $f_1=f_2$. Since we concluded that the coefficients are equal one can alternatively use [19, Theorem 1] to obtain uniqueness of the sources.

The obtained results can be summarised as follows.

Theorem 5. *Let the spacewise conductivity coefficient and heat source $(a,f)\in\mathcal{A}$, where $\mathcal{A}$ is given by (7), with the coefficient a satisfying Assumption 1. Moreover, assume that the initial condition h and the final time measurement g are regular enough such that the corresponding solution $u(x,t)$ of (1) satisfies*

$$\|\nabla u_t\|_{L^\infty((0,T)\times\Omega)} + \|\triangle u_t\|_{L^\infty((0,T)\times\Omega)} < M.$$

Moreover, assume that the final time condition g satisfies the Assumption (21) in Lemma 4 and (25). Then the inverse problem of identifying $\{a(x), f(x), u(a,f)\}$ in the heat equation (1) for a given additional measurement $g(x)=u(x,T)$ has a unique solution.

In the Appendix we shall prove that there exist conductivities and sources which can generate a final time value satisfying (21), i.e. the inverse problem (1)–(2) with (21) imposed will have a solution for some data (and this solution is unique according to the above theorem).

We could now dwell into lengthy calculations on the uniqueness of the inverse problem (1)–(2) under less regularity assumptions on the coefficient and data. There are indeed many generalizations of Carleman estimates for parabolic equations that one could potentially use in this case similar to the arguments in [18, 17, 20, 21]. However, we prefer not to enter into these technicalities.

Let us though briefly discuss some alternative estimates in the above proof of uniqueness that could potentially reduce the imposed smootheness assumptions. We used the estimate (13) to obtain an upper bound of the right-hand side of (12). However, there are alternative estimates that can be used. For example, in [18, Subsection 3.2] the right-hand side of (12) is estimated by

$$\int_Q |\nabla v_t|^2 |a-b|^2 e^{-2s\varphi}\, dxdt\,.$$

Note that, with the above estimate, equation (15) follows with less assumptions on the regularity of the coefficients as well as of (1). Moreover, in [18, Theorem 3.1] a Carleman estimate (for $L^\infty(\Omega)$ coefficients) relating the L^2-norm of the coefficients with the right-hand side of equation (16) is given. In other words, [18, Thoerem 3.1] implies that one do not need to differentiate as high in time as we did in the second equation of (16). These two factors together would, most likely, improve and weaken the smoothness assumptions in Theorem 5.

We point out that the Assumption 1 was imposed merely for technical reasons to simplify the presentation of the Carleman estimates and to avoid cut-off functions. To convince the reader that this assumption can be removed and to further highlight that our assumptions are far from the most general ones, we discuss in the next section the uniqueness for a more general class of spacewise parameters in the one-dimensional parabolic heat equation using a completely different technique, which does not lend itself to higher dimensions though.

3 Uniqueness of the spacewise heat conductivity and heat source in a one-dimensional heat equation

In this section we show the uniqueness under weaker smoothness assumptions compared with Section 2 of the identification of the spacewise pair of coefficient and source (a, f) in (1), with additional measurements given by (2). In particular, the Assumption 1 will not be used. However, we only consider the case with $\Omega = (0, L) \subset \mathbb{R}$, i.e. the spacewise solution domain in (1)–(2) is one-dimensional in space.

The derivation of the uniqueness result is based on a completely different technique than Carleman estimates. Indeed, the technique is based on results that relate the uniqueness of the inverse identification problem to the density in certain function spaces of solutions of the corresponding adjoint problem. With this approach the assumptions on the smoothness of the unknown pair (a, f) is reduced to a more general class. Moreover, assumptions on the smoothness of the input and measured data are determined only by extracting a certain differential dependence and that there exists a solution for the corresponding direct problem.

We start by assuming the following regularity conditions for the parameter, the source, the initial condition and the measured data in the inverse problems (1)–(2); compare with the assumptions in Section 2.

Assumption 6. *We assume that the heat conductivity* $a \in L^\infty((0, L))$ *and that there exists* $\underline{a} > 0$ *such that* $a(x) \geq \underline{a}$ *for all* $x \in (0, L)$. *Moreover, it is assumed that* $a(\cdot)$ *is continuous in* $[0, \varepsilon)$ *and in* $(L - \varepsilon, L]$ *for any fixed* $\varepsilon > 0$, *the heat source* $f \in L^2([0, L])$, *the initial temperature* $h \in H^1([0, L])$, *and that the additional final time temperature measurement* $g \in H^1([0, L])$.

Since $h, g \in H^1([0, L])$ we further need to assume that the following matching conditions (compatibility) are satisfied in $x = 0$ and $x = L$

$$
\begin{aligned}
-(a(0)h_x(0))_x &= f(0)\,, \\
-(a(L)h_x(L))_x &= f(L)\,, \\
-(a(0)g_x(0))_x &= f(0)\,, \\
-(a(L)g_x(L))_x &= f(L)\,.
\end{aligned} \tag{27}
$$

For the existence and regularity of a solution for the corresponding direct problem with the pair $(a(x), f(x))$ satisfying the conditions on regularity stated in Assumption 6, we have:

Lemma 7. *Let the Assumption 6 and the matching conditions (27) hold. Then, there exists a unique solution $u(x,t) \in H^{2,1}([0,L] \times [0,T])$ of the boundary value problem (1). Moreover, there exists a constant M_0 such that $\|u\|_{C([0,L]\times[0,T])} \le M_0$.*

Proof. The existence and uniqueness follows directly from classical results on parabolic partial differential equations, see for example [16, 15]. Now, from the Sobolev embedding Theorem [15, 16, 1] we have that $u(x,t) \in C([0,L] \times [0,T])$. The uniform boundedness follows from the maximum principle for parabolic equations [15] together with the assumed smoothness of the boundary, initial and final data. □

3.1 Uniqueness of a solution of the inverse problem: 1-d case

The steps for proving uniqueness of the identification of the pair of parameters $\{a(x), f(x)\}$ for given initial and final data in (1)–(2) are outlined below:

Assume that $u = u(a, f_1)$ and $v = u(b, f_2)$ are two solutions of (1) with additional data (2). As in the previous section let $w = u - v$. Then, w satisfies

$$
w_t - (a(x)w_x)_x = ([a(x) - b(x)]v_x)_x + (f_1(x) - f_2(x)) \text{ in } (0,L) \times (0,T) \tag{28}
$$

with homogeneous initial, boundary and final conditions.

For the proof uniqueness in the one-dimensional case we shall invoke the adjoint problem of (28), that reads as

$$
\begin{aligned}
\psi_t + (a(x)\psi_x)_x &= 0 \text{ in } (0,L) \times (0,T) \\
\psi(0,t) = \psi(L,t) &= 0 \text{ for } t \in (0,T) \\
\psi(x,0) &= 0 \text{ for } x \in (0,L) \\
\psi(x,T) &= \mu(x) \text{ for } x \in (0,L),
\end{aligned} \tag{29}
$$

where $\mu(x)$ is an arbitrary function in $C_0^2[0,L]$.

For properties of solutions of the adjoint equation (29) we have:

Lemma 8. *Let the Assumption 6 hold.*

i) For any function $\mu(x) \in C_0^2[0,L]$, there exists a unique solution $\psi(x,t;v) \in C^1((0,T); C^2(0,L)) \cap C([0,L] \times [0,T])$ of (29).

ii) For any function $\mu(x) \in C_0^2[0, L]$, the following relation holds

$$\int_0^T \int_0^L \psi(x,t;\mu(x))F(x,t)\,dxdt = 0, \tag{30}$$

where w is a solution to (28) with right-hand side $F(x,t)$.

iii) For $\mu(x)$ ranging over the space $C_0^2[0, L]$, the corresponding range of $\psi(x,t;\mu(x))|_{t=\tau}$ is everywhere dense in $L^2[0, L]$ for any time $t = \tau$, $0 \le \tau \le T$.

iv) Given that

$$\int_0^T \int_0^L \psi(x,t;\mu(x))\Phi(x,t)\,dxdt = 0$$

for $\mu(x)$ ranging over the space $C_0^2[0, L]$, then

$$\Phi(x,T) = 0.$$

Proof. Item i) is a well-known result for parabolic equations, see, for example, [16, 15]. Item ii) follows immediately by multiplication of (32) by ψ and integration by parts. Item iii) and Item iv) are consequences of [5, Lemma 2-3, pg 318].

□

We now have the required results in order to prove the main step in the uniqueness argument for the conductivity function $a(x)$ by adding some additional assumptions on the final data in (1)–(2).

Theorem 9. *Let the Assumption 6 and the matching condition (27) hold. Moreover, assume that $|g_x(x)| > 0$ for all $0 \le x \le L$. Then the inverse problem (1)–(2) has a unique solution $\{u, a, f\}$ with the conductivity $a \in L^\infty(0, L)$, the heat source $f \in L^2(0, L)$, and temperature u, with $\|u\|_1 < \infty$.*

Proof. Since we have time-independent coefficients and source, it follows that the solution v to (1) with additional data (2) has derivatives of all orders with respect to t [9, 20] and this in turn implies that w has derivatives of all orders with respect to t. Moreover, due to the Sobolev imbedding theorem, we can assume that w in (28) is at least continuous; for simplicity we assume that pointwise evaluation makes sense for the coefficient and source.

Note that at $x = 0$ and at $x = L$ using the assumptions and matching condition (27) imply that

$$a(x) = b(x) \quad \text{for } x = 0 \text{ and } x = L\,. \tag{31}$$

In order to prove that $a(x) = b(x)$ for $0 < x < L$ we apply Lemma 8 Item ii) in combination with iv) in (30) to get

$$[(a(x) - b(x))(v(x,T))_x]_x + (f_1(x) + f_2(x)) = 0$$

for $0 \le x \le L$. Using this in the first equation in (28) in combination with $w(x,T) = 0$ for every x in $[0, L]$, we conclude that $w_t(x,T) = 0$.

Define $z = w_t$. Similar to the previous section we have that z satisfies

$$\begin{aligned} z_t - (a(x)z_x)_x &= ([a(x) - b(x)](v_x)_t)_x \text{ in } \ (0,L)\times(0,T) \\ z(0,t) = z(L,t) &= 0 \text{ for } t \in (0,T) \\ z(x,0) &= \theta_1(x) \text{ for } x \in (0,L) \\ z(x,T) &= 0 \text{ for } x \in (0,L). \end{aligned} \tag{32}$$

Splitting this problem into two, one with zero right-hand side and with initial condition θ_1 and one with the given right-hand side and zero initial condition, following the proof of Lemma 2 one can conclude that the solution to the first one is identically zero, i.e. $\theta_1(x) = 0$. Therefore, since z satisfies a problem of the same kind as w we can again apply Lemma 8 Item ii) in combination with iv) in (30) to conclude that

$$[(a(x) - b(x))(v_t(x,T))_x]_x = 0.$$

Using this in the first equation in (32) in combination with $z(x,T) = 0$ for every x in $[0,L]$, we conclude that $z_t(x,T) = 0$, i.e. $w_{tt}(x,T) = 0$. Continuing this, putting $z_1 = z_t$ and deriving the problem for z_1 and applying the similar reasoning, i.e. Lemma 8 Item ii) in combination with iv), we find that $w_{ttt}(x,T) = 0$. Further continuing this it is possible to prove that $(\partial_t^{(k)} w)(x,T) = 0$ for $k = 0,1,2,\dots$ and the same holds at $t = 0$. From this and strong unique continuation results for parabolic equations, we conclude that $w(x,t) = 0$ for $[0,L]\times[0,T]$. This in particular implies that $z = 0$ in $[0,L]\times[0,T]$. From the first equation in (32) we then have

$$[(a(x) - b(x))(v_t(x,t))_x]_x = 0$$

for every (x,t) in $[0,L]\times[0,T]$. Integrating first with respect to x using that $a(0) = b(0)$, we find that

$$(a(x) - b(x))(v_t(x,t))_x = 0.$$

Since the coefficients are independent of time, we write this as

$$[(a(x) - b(x))(v_x(x,t))]_t = 0.$$

Integrating with respect to time using $a(0) = b(0)$, and then putting $t = T$ we obtain

$$(a(x) - b(x))g_x(x) = 0$$

for $0 \le x \le L$. From the assumptions on g we can conclude that $a(x) = b(x)$ also for $0 < x < L$.

The final step in the uniqueness argument is the proof of unique identifiability of $f(x)$ in (1)–(2). Since $w = 0$ and $a = b$, we have from (28) that

$$f_1(x) - f_2(x) = 0$$

for every x in $[0,L]$, i.e. $f_1 = f_2$ and the theorem is proved. □

We finish remarking that the main uniqueness argument is related to the thermal diffusivity coefficient a. However, the arguments that we have presented in this section appear not possible to generalize to dimensions in space higher than one. This is due to the fact that we would obtain an equation where the divergence of an element is zero. However, we can not conclude that the given element would then be a constant, as the simple example $(x,-y)$ shows.

4 Conclusions and possible generalizations

In this paper, we proved uniqueness for the inverse problem of simultaneously identifying a spacewise heat conductivity and heat source for a given final time measurement. The main result is based on local Carleman estimates for parabolic problems following [21]. We did not strive for the most general result, but only aimed at convincing the reader that there can be at most one coefficient and source that satisfy a given final time condition. Therefore, we assumed rather strong regularity on the unknown parameters and worked with classical solutions. However, these can be relaxed and it was indicated how to adjust the proof. For conductivities and sources in spaces of integrable functions an alternative proof of uniqueness in the one-dimensional case was given, where many of the assumptions on smoothness were weakend. Unfortunately, these arguments for the one-dimensional case can not be generalized to dimensions higher than one. The one-dimensional case does motivate the uniqueness of the inverse problem under less regularity assumptions on the parameter spaces; to present such a result in higher dimension is deferred to future work.

We remark that there are actually many strong results on Carleman estimates for parabolic equations with very general assumptions on the smoothness of the parameters [18, 17]. Moreover, we used local Carleman estimates in order to simplify the proof of uniqueness in the higher-dimensional case. However, we may use global Carleman estimates to avoid the assumption that the heat conductivity is known close to the boundary of the body Ω, see [13, 17, 18, 20, 21] for a general overview on the subject.

Therefore, the authors' conjecture is that there exists uniqueness of the spacewise heat conductivity and heat source for a given final time additional measurement, under weaker assumptions on the parameter space. It will be addressed in a future works together with a regularization method for the inverse problem.

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A Appendix

In this appendix, we show that the assumption (21) on the final time measurement (2) required in Section 2, can be satisfied, i.e. the inverse problem (1)–(2) with (21) imposed can have a solution for a given initial data being sufficiently regular and with a localized heat source.

The first result is on the regularity of the solution of (1). For simplicity, we assume that all compatibility conditions are satisfied and that Ω is an open, bounded and regular subset of $\mathbb{R}^n$. We do not present a proof of the following theorem, since it follows from standard regularity estimates for parabolic equations. Moreover, since the coefficient and the source are spacewise dependent, the solution is analytic in time. For details see, for example, [16, 15].

Theorem A.1. *Let the coefficient and the source belong to the admissible set $\mathcal{A}$. Then, for any $h \in H^k(\Omega)$, there exists a unique solution $u \in C([0,T]; H^{k+1}(\Omega)) \cap C((0,T); H^1_0(\Omega) \cap H^{k+1}(\Omega)) \cap$*

$C^1((0,T);H^{k+1}(\Omega))$ *of the parabolic equation* (1).

Note that the regularity above is far from optimal.

Lemma A.2. *Let* $k > \max\{3, n/2\}$. *Then the assumption about the finiteness of the constant* C *in* (15) *and assumption* (25) *are satisfied for the solution of* (1) *with additional data* (2).

Proof. This follows from Theorem A.1 and the continuous embedding of $H^k(\Omega) \cap H_0^1(\Omega)$ in $C(\overline{\Omega})$, [1]. ☐

Now, we verify Assumption 21 in Lemma 4. We remark that Lemma 4 is a corollary of [21, Lemma 6.2] that reads as follows

Proposition A.3. *Let the solution domain* Ω *be as above. Put* $\zeta = a - b$. *Consider the first-order partial differential equation*

$$(P_0\zeta)(x) = \nabla\zeta(x) \cdot \nabla g(x) + \zeta(x) \triangle g(x)$$

where $g \in H^k(\Omega)$, *for* k *as in Lemma A.2.*

If the Carleman weight function $\varphi(x,T) \in C^1(\overline{\Omega})$ *satisfies*

$$\nabla g(x) \cdot \nabla\varphi(x,T) > 0\,, \quad x \in \overline{\Omega}\,, \tag{A.1}$$

then the Carleman estimate (22) *is satisfied.*

Therefore, we only need to prove that there exists a Carleman weight function φ, a set of initial data and a heat source such that (A.1) is satisfied. However, by construction of the Carleman weight function, we have

$$\nabla\varphi(x,T) = \nabla d(x) e^{\lambda\beta(x,T)}\,.$$

Hence, is enough to guarantee that there exist $d(x)$ such that

$$\nabla g(x) \cdot \nabla d(x) > 0\,, \quad x \in \overline{\Omega}\,. \tag{A.2}$$

It is verified in the following lemma.

Lemma A.4. *Let* $\varepsilon > 0$ *and let* $\mathcal{O}$ *be any open set of* Ω. *There exists* $d \in C(\overline{\Omega})$ *with* $d|_{\partial\Omega} = 0$ *and* $|\nabla d(x)| \geq \varepsilon$ *for* $x \in \Omega$, *such that for any* $h \in L^2(\Omega)$ *there is a sufficiently smooth source term* f, *having support in* $\mathcal{O}$, *with*

$$\|\nabla u(f,h)(T) - \nabla d\|_{L^\infty(\Omega)} < \varepsilon/\sqrt{2}\,, \tag{A.3}$$

where $u(f,h)$ *is the solution of* (1) *with source* f *and initial data* h. *Moreover,* (A.2) *holds.*

Proof. Since we do not use global Carleman estimates, we can consider d to be zero on $\partial\Omega$. For the existence of such a function d, see for example [18, Section 2]. The density argument then follows from [22, Proposition 1.1], see also [18, Corollary 3.1].

Hence, from $g(x) = u(f,h)(T)$, the regularity of $g(x)$ and $d(x)$ and the estimate (A.3), we have

$$\varepsilon^2/2 > (ess\sup|\nabla g(x) - \nabla d(x)|)^2 \geq |\nabla g(x) - \nabla d(x)|^2 = |\nabla g(x)|^2 - 2\nabla g(x)\cdot\nabla d(x) + |\nabla d(x)|^2\,.$$

Therefore, since $|\nabla d| \geq \varepsilon$ it follows that

$$2\nabla g(x)\cdot\nabla d(x) > |\nabla g(x)|^2 + |\nabla d(x)|^2 - \varepsilon^2/2 \geq |\nabla d(x)|^2 - \varepsilon^2/2 \geq \varepsilon^2/2\,.$$

☐

The condition (A.2) can then be stated as (21) via a suitable transformation, see the proof of [21, Lemma 6.1]. Thus, we have shown that there are conductivities and sources that can generate a final time value such that (21) holds, i.e. the inverse problem (1)–(2) with (21) imposed will have a solution for some data (and this solution is unique). Again, we have not investigated optimal conditions and there might be other restrictions on the final data that generate uniqueness. Note that the condition (21) is similar to the condition imposed in the one- dimensional case in Section 3.

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