On a level-set method for ill-posed problems with piecewise non-constant coefficients

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Abstract

We investigate a level-set type method for solving ill-posed problems, with the assumption that the solutions are piecewise, but not necessarily constant functions with unknown level sets and unknown level values. In order to get stable approximate solutions of the inverse problem we propose a Tikhonov-type regularization approach coupled with a level set framework. We prove the existence of generalized minimizers for the Tikhonov functional. Moreover, we prove convergence and stability for regularized solutions with respect to the noise level, characterizing the level-set approach as a regularization method for inverse problems. We also show the applicability of the proposed level set method in some interesting inverse problems arising in elliptic PDE models.

Keywords: Level Set Methods, Regularization, Ill-Posed Problems, Piecewise Non-Constant Coefficients.

1 Introduction

Since the seminal paper of Santosa [30], level set techniques have been successfully developed and have recently become a standard technique for solving inverse problems with interfaces (e.g., [5, 8, 11, 17, 18, 27, 29, 36, 37]).

In many applications, interfaces represent interesting physical parameters (inhomogeneities, heat conductivity between materials with different heat capacity, interface diffusion problems) across which one or more of these physical parameters change value in a discontinuous manner. The interfaces divide the domain $\Omega \subset \mathbb{R}^n$ in subdomains Ω_j, with $j = 1, \cdots, k$, of different regions with specific internal parameter profiles. Due to the different physical structures of each of these regions, different mathematical models might be the most appropriate for describing them. Solutions of such models represent a free boundary problem, i.e., one in which interfaces are also unknown and must be determined in addition to the solution of the governing partial differential equation. In general such solutions are determined by a set of data obtained by indirect measurements [5, 8, 9, 10, 11, 26, 35, 42]. Applications include image segmentation problems [10, 26, 35, 42], optimal shape designer problems [32, 5], Stefan's type problems [5], inverse potential problems [15, 13, 14], inverse conductivity/resistivity problems [9, 11, 17, 24, 37] among others [5, 8, 11, 18, 32].

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There is often a large variety of priors information available for determining the unknown physical parameter, whose characteristic depends on the given application. In this article, we are interested in inverse problems that consist in the identification of an unknown quantity $u \in D(F) \subset X$ that represents all parameter profiles inside the individual subregions of Ω, from data $y \in Y$, where X and Y are Banach spaces and $D(F)$ will be adequately specified in Section 3. In this particular case, only the interfaces between the different regions and, possibly, the unknown parameter values need to be reconstructed from the gathered data. This process can be formally described by the operator equation

$$F(u) \;=\; y\,, \tag{1}$$

where $F : D(F) \subset X \to Y$ is the forward operator.

Neither existence nor uniqueness of a solution to (1) are guarantee. For simplicity, we assume that for exact data $y \in Y$, the operator equation (1) admit a solution and we do not strive to obtain results on uniqueness. However, in practical applications, data are obtained only by indirect measurements of the parameter. Hence, in general, exact data $y \in Y$ are not known and we have only access to noise data $y^\delta \in Y$, whose level of noise $\delta > 0$ are assumed be known *a priori* and satisfies

$$\|y^\delta - y\|_Y \;\leq\; \delta\,. \tag{2}$$

We assume that the inverse problem associated with the operator equation (1) is ill-posed. Indeed, it is the case in many interesting problems [11, 18, 19, 23, 24, 32, 37]. Therefore, accuracy of an approximated solution call for a regularization method [19]. In this article we propose a Tikhonov-type regularization method coupled with a level-set approach to obtain a stable approximation of the unknown level sets and values of the piecewise (not necessarily constant) solution of (1).

Many approaches, in particular level set type approaches, have previously been suggested for such problems . In [6, 7, 9, 22, 25, 30], level set approaches for identification of the unknown parameter u with distinct, but known, piecewise constant values were investigated.

In [7, 10, 15], level set approaches were derived to solve inverse problems, assuming that u is defined by several distinct constant values. In both cases, one needs only to identify the level sets of u, i.e. the inverse problem reduces to a shape identification problem. On the other hand, when the level values of u are also unknown, the inverse problem becomes harder, since, we have to identify both the level sets and the level values of the unknown parameter u. In this situation, the dimension of the parameter space increases by the number of unknown level values. Level set approaches to ill- posed problems with unknown constant level values appeared before in [14, 13, 32, 33, 35]. Level set regularization properties of the approximated solution for inverse problems are described in [4, 15, 13, 14, 22].

However, regularization theory for inverse problems where the components of the parameter u are variable and have discontinuities have not been well investigated. Indeed, level set regularization theory applied to inverse problems [13, 15, 14] that recover the shape and the values of variable discontinuous coefficients are unknown to the author. Some early results in the numerical implementation of level set type methods were previously used to obtain solutions of elliptic problems with discontinuous and variable coefficients in [11].

In this article, we propose a level set type regularization method to ill-posed problems whose solution is composed by piecewise components which are not necessarily constants. In other words, we introduce a level set type regularization method to recover the shape and the values of variable discontinuous coefficients. In this framework a level set function is used to parameterized the solution u of (1). We obtain a regularized solution using a Tikhonov-type regularization method. Since the level values of u are not-constant and also unknown.

In the theoretical point of view, the advantage of our approach in relation to [5, 14, 15, 13, 22, 38] is that we are able to obtain regularized solutions to inverse problems with piecewise solutions that are more general than those covered by the regularization methods proposed before. We still prove regularization properties for the approximated solution of the inverse problem model (1), where the parameter is a non-constant piecewise solution. The topologies needed to guarantee the existence of a minimizer (in a generalized sense) of the Tikhonov functional (define below in (5)) is quite complicated and differ in some key points from [13, 14, 22]. In this particular approach, the definition of generalized minimizers are quite different from other works [14, 15, 22] (see Definition 1). As a consequence, the arguments used to prove the well-posedeness of the Tikhonov functional, the stability and convergence of the regularized solutions of the inverse problem (1) are quite complicated and need significant improvements (see Section 3).

The main applicability advantage of the proposed level set type method compared to those in the literature is that we are able to apply this method to problems whose solutions depend of non-constant parameters. This implies that we are able to handle more general and interesting physical problems, where de components of the desired parameter is not necessarily homogeneous, as those presented before in the literature [3, 11, 18, 14, 13, 32, 33, 35, 34, 40]. Examples of such interesting physical problems are heat conduction between materials of different heat capacity and conductivity, interface diffusion processes and many other types of physical problems where modeling components are related with embedded boundaries. See for example [3, 14, 8, 11, 18, 40] and references therein. As a benchmark problem we analyze two inverse problems modeled by elliptic PDE's with discontinuous and variable coefficients.

In contrast, the non-constant characteristics of the level values impose different types of theoretical problems, since the topologies where we are able to provide regularization properties of the approximated solution are more complicated than the ones presented before [14, 13, 32, 33, 35]. As a consequence, the numerical implementations becomes harder than the others approaches in the literature [14, 13, 40, 38].

The paper is outlined as follows: In Section 2, we formulate the Tikhonov functional based on the level-set framework. In Section 3, we present the general assumptions needed in this article and the definition of the set of admissible solutions. We prove relevant properties about the admissible set of solutions, in particular, convergence in suitable topologies. We also present relevant properties of the penalization functional. In Section 4, we prove that the proposed method is a regularization method to inverse problems, i.e., we prove that the minimizers of the proposed Tikhonov functional are stable and convergent with respect to the noise level in the data. In Section 5, a smooth functional is proposed to approximate minimizers of the Tikhonov functional defined in the admissible set of solutions. We provide approximation properties and the optimality condition for the minimizers of the smooth Tikhonov functional. In Section 6, we present an application of the proposed framework to solve some interesting inverse elliptic problems with variable coefficients. Conclusions and future directions are presented in Section 7.

2 The Level-set Formulation

Our starting point is the assumption that the parameter u in (1) assumes two unknown functional values, i.e., $u(x) \in \{\psi^1(x), \psi^2(x)\}$ a.e. in $\Omega \subset \mathbb{R}^d$, where Ω is a bounded set. More specifically, we assume the existence of a mensurable set $D \subset\subset \Omega$, with $0 < |D| < |\Omega|$, such that $u(x) = \psi^1(x)$ if $x \in D$ and $u(x) = \psi^2(x)$ if $x \in \Omega/D$. With this framework, the inverse problem that we are interested in this article is the stable identification of both the shape of D and the value function $\psi^j(x)$ for x belonging to D and to Ω/D, respectively, from observation of the data $y^\delta \in Y$.

We remark that, if $\psi^1(x) = c^1$ and $\psi^2(x) = c^2$ with c^1 and c^2 unknown constants values, the problem of identifying u was rigorously studied before in [14]. Moreover, many other approaches to this case appear in the literature, see [14, 7, 6, 5] and references therein. Recently, in [13], a L^2 level set approach to identify the level and constant contrast was investigated.

Our approach differs from the level set methods proposed in [13, 14], by considering also the identification of variable unknown levels of the parameter u. In this situation, many topological difficulties appear in order to have a tractable definition of an admissible set of parameters (see Definition 1 below). Generalization to problems with more than two levels are possible applying this approach and following the techniques derived in [15]. As observed before, the present level set approach is a rigorous derivation of a regularization strategy for identification of the shape and non-constant levels of discontinuous parameters. Therefore, it can be applied to physical problems modeled by embedded boundaries whose components are not necessarily piecewise constant [22, 5, 15, 13, 14].

In many interesting applications, the inverse problem modeled by equation (1) is ill-posed. Therefore a regularization method must be applied in order to obtain a stable approximate solution. We propose a regularization method by: First, introduce a parametrization on the parameter space, using a level set function ϕ that belongs to $H^1(\Omega)$. Note that, we can identify the distinct level sets of the function $\phi \in H^1(\Omega)$ with the definition of the Heaviside projector

$$
\begin{aligned}
H : H^1(\Omega) &\longrightarrow L^\infty(\Omega) \\
\phi \quad &\longmapsto \quad H(\phi) := \begin{cases} 1 & \text{if} \quad \phi(x) > 0\,, \\ 0 & \text{other else}\,. \end{cases}
\end{aligned}
$$

Now, from the framework introduced above, a solution u of (1), can be represented as

$$
u(x) \;=\; \psi^1(x) H(\phi) + \psi^2(x)(1 - H(\phi)) \;=:\; P(\phi, \psi^1, \psi^2)(x)\,. \tag{3}
$$

With this notation, we are able to determine the shapes of D as $\{x \in \Omega\,;\ \phi(x) > 0\}$ and Ω/D as $\{x \in \Omega\,;\ \phi(x) < 0\}$.

The functional level values $\psi^1(x)$, $\psi^2(x)$ are also assumed be unknown and they should be determined as well.

Assumption 1. *We assume that* $\psi^1, \psi^2 \in \mathbb{B} := \{f : f \text{ is measurable and } f(x) \in [m, M]\,, \text{a.e. in } \Omega\}$, *for some constant values* m, M.

Remark 1. *We remark that,* $f \in \mathbb{B}$ *implies that* $f \in L^\infty(\Omega)$. *Since* Ω *is bounded* $f \in L^1(\Omega)$. *Moreover,*

$$
\int_\Omega f(x) \nabla \cdot \varphi(x) dx \le |M| \int_\Omega |\nabla \cdot (\varphi)(x)| dx \le |M| \|\nabla \cdot \varphi\|_{L^1(\Omega)}\,, \forall \varphi \in C_0^1(\Omega, \mathbb{R}^n)\,.
$$

Hence $f \in \mathsf{BV}(\Omega)$.

Note that, in the case that ψ^1 and ψ^2 assumes two distinct constant values (as covered by the analysis done in [5, 14, 13] and references therein) the assumptions above are satisfied. Hence, the level set approach proposed here generalizes the regularization theory developed in [14, 13].

From (3), the inverse problem in (1), with data given as in (2), can be abstractly written as the operator equation

$$
F(P(\phi, \psi^1, \psi^2)) \;=\; y^\delta\,. \tag{4}
$$

Once an approximate solution (ϕ, ψ^1, ψ^2) of (4) is obtained, a corresponding solution of (1) can be computed using equation (3).

Therefore, to obtain a regularized approximated solution to (4), we shall consider the least square approach combined with a regularization term i.e., minimizing the Tikhonov functional

$$\hat{\mathcal{G}_\alpha}(\phi, \psi^1, \psi^2) := \|F(P(\phi, \psi^1, \psi^2)) - y^\delta\|_Y^2 + \alpha\Big\{\beta_1|H(\phi)|_{\mathtt{BV}} + \beta_2\|\phi - \phi_0\|_{H^1(\Omega)}^2 + \beta_3\sum_{j=1}^{2}|\psi^j - \psi_0^j|_{\mathtt{BV}}\Big\}, \tag{5}$$

where, ϕ_0 and ψ_0^j represent some *a priori* information about the true solution u^* of (1). The parameter $\alpha > 0$ plays the role of a regularization parameter and the values of $\beta_i\,, i = 1, 2, 3$ act as scaling factors. In other words, $\beta_i\,, i = 1, 2, 3$ need to be chosen *a priori*, but independent of the noise level δ. In practical, $\beta_i\,, i = 1, 2, 3$ can be chosen in order to represent *a priori* knowledge of features the of the parameter solution u and/or to improve the numerical algorithm. A more complete discussion about how to choose $\beta_i\,, i = 1, 2, 3$ are provided in [13, 14, 15].

The regularization strategy in this context is based on $TV - H^1 - TV$ penalization. The term on H^1-norm acts simultaneously as a control on the size of the norm of the level set function and a regularization on the space H^1. The term on $\mathtt{BV}$ is a variational measure of $H(\phi)$. It is well known that the BV-semi-norm acts as a penalizing for the length of the Hausdorff measure of the boundary of the set $\{x : \phi(x) > 0\}$ (see [21, Chapter 5] for details). Finally, the last term on $\mathtt{BV}$ is a variational measure of ψ^j that acts as a regularization term on the set $\mathbb{B}$. This Tikhonov functional extends the ones proposed in [14, 15, 6, 7, 32] (based on TV-H^1 penalization).

Existence of minimizers for the functional (5), in the $H^1 \times \mathbb{B}^2$ topology does not follow by direct arguments, since, the operator P is not necessarily continuous in this topology. Indeed, if $\psi^1 = \psi^2 = \psi$ is a continuous function at the contact region, then $P(\phi^1, \psi^2, \psi) = \psi$ is continuous and the standard Tikhonov regularization theory to the inverse problem holds true [19]. On the other hand, in the interesting case where ψ^1 and ψ^2 represents the level of discontinuities of the parameter u, the analysis becames more complicated and we need a definition of generalized minimizers (see Definition 1) in order to handle with these difficulties.

3 Generalized Minimizers

As already observed in [22], if $D \subset \Omega$ with $\mathcal{H}^{n-1}(\partial D) < \infty$ where $\mathcal{H}^{n-1}(S)$ denotes the (n-1)-dimensional Hausdorff-measure of the set S, then the Heaviside operator H maps $H^1(\Omega)$ into the set

$$\mathcal{V} := \{\chi_D\,;\ D \subset \Omega \text{ measurable}, :\ \mathcal{H}^{n-1}(\partial D) < \infty\}\,.$$

Therefore, the operator P in (3) maps $H^1(\Omega) \times \mathbb{B}^2$ into the admissible parameter set

$$D(F) := \{u = q(v, \psi^1, \psi^2)\,;\ v \in \mathcal{V} \text{ and } \psi^1, \psi^2 \in \mathbb{B}\}\,,$$

where

$$q : \mathcal{V} \times \mathbb{B}^2 \ni (v, \psi^1, \psi^2) \mapsto \psi^1 v + \psi^2(1 - v) \in BV(\Omega)\,.$$

Consider the model problem described in the introduction. In this article, we assume that:

(A1) $\Omega \subseteq \mathbb{R}^n$ is bounded with piecewise C^1 boundary $\partial\Omega$.
(A2) The operator $F : D(F) \subset L^1(\Omega) \to Y$ is continuous on $D(F)$ with respect to the $L^1(\Omega)$-topology.
(A3) ε, α and β_j , $j = 1, 2, 3$ denote positive parameters.
(A4) Equation (1) has a solution, i.e. there exists $u_* \in D(F)$ satisfying $F(u_*) = y$ and a function $\phi_* \in H^1(\Omega)$ satisfying $|\nabla\phi_*| \neq 0$, in the neighborhood of $\{\phi_* = 0\}$ such that $H(\phi_*) = z_*$, for some $z_* \in \mathcal{V}$. Moreover, there exist functional values $\psi^1_*, \psi^2_* \in \mathbb{B}$ such that $q(z_*, \psi^1_*, \psi^2_*) = u_*$.

For each $\varepsilon > 0$, we define a smooth approximation to the operator P by

$$P_\varepsilon(\phi, \psi^1, \psi^2) \ := \ \psi^1 H_\varepsilon(\phi) + \psi^2(1 - H_\varepsilon(\phi)) \, , \tag{6}$$

where H_ε is the smooth approximation to H described by

$$H_\varepsilon(t) := \begin{cases} 1 + t/\varepsilon & \text{for} \ \ t \in [-\varepsilon, 0] \\ \quad H(t) & \text{for} \ \ t \in \mathbb{R}/[-\varepsilon, 0] \end{cases} .$$

Remark 2. *It is worth noting that, for any $\phi_k \in H^1(\Omega)$, $H_\varepsilon(\phi_k)$ belongs to $L^\infty(\Omega)$ and satisfies $0 \le H_\varepsilon(\phi_k) \le 1$ a.e. in Ω, for all $\varepsilon > 0$. Moreover, taking into account that $\psi^j \in \mathbb{B}$, follows that the operators q and P_ε, as above, are well defined.*

In order to guarantee the existence of a minimizer of $\mathcal{G}_\alpha$ defined in (5) in the space $H^1(\Omega) \times \mathbb{B}^2$, we need to introduce a suitable topology such that the functional $\mathcal{G}_\alpha$ has a closed graphic. Therefore, the concept of generalized minimizers (compare with [15, 22]) in this paper is:

Definition 1. *Let the operators H, P, H_ε and P_ε be defined as above and the positive parameters α, β_j and ε satisfying the Assumption **(A3)**.*

A **quadruple** $(z, \phi, \psi^1, \psi^2) \in L^\infty(\Omega) \times H^1(\Omega) \times \mathtt{BV}(\Omega)^2$ *is called* **admissible** *when:*
a) There exists a sequence $\{\phi_k\}$ of $H^1(\Omega)$-functions satisfying $\lim\limits_{k\to\infty} \|\phi_k - \phi\|_{L^2(\Omega)} = 0$.
b) There exists a sequence $\{\varepsilon_k\} \in \mathbb{R}^+$ converging to zero such that $\lim\limits_{k\to\infty} \|H_{\varepsilon_k}(\phi_k) - z\|_{L^1(\Omega)} = 0$.
c) There exist sequences $\{\psi^1_k\}_{k\in\mathbb{N}}$ and $\{\psi^2_k\}_{k\in\mathbb{N}}$ belonging to $\mathtt{BV} \cap C^\infty(\Omega)$ such that

$$|\psi^j_k|_{\mathtt{BV}} \longrightarrow |\psi^j|_{\mathtt{BV}} \, , \quad j = 1, 2 \, .$$

d) A **generalized minimizer** *of $\hat{\mathcal{G}}_\alpha$ is considered to be any admissible quadruple $(z, \phi, \psi^1, \psi^2)$ minimizing*

$$\mathcal{G}_\alpha(z, \phi, \psi^1, \psi^2) := \left\| F(q(z, \psi^1, \psi^2)) - y^\delta \right\|_Y^2 + \alpha R(z, \phi, \psi^1, \psi^2) \tag{7}$$

on the set of admissible quadruples. Here the functional R is defined by

$$R(z, \phi, \psi^1, \psi^2) \ = \ \rho(z, \phi) + \beta_3 \sum_{j=1}^{2} |\psi^j - \psi^j_0|_{\mathtt{BV}} \tag{8}$$

and the functional ρ is defined as

$$\rho(z, \phi) := \inf \left\{ \liminf_{k\to\infty} \left[\beta_1 |H_{\varepsilon_k}(\phi_k)|_{\mathtt{BV}} + \beta_2 \|\phi_k - \phi_0\|^2_{H^1(\Omega)} \right] \right\} . \tag{9}$$

The infimum in (9) is taken over all sequences $\{\varepsilon_k\}$ and $\{\phi_k\}$ characterizing $(z, \phi, \psi^1, \psi^2)$ as an admissible quadruple.

The convergence $|\psi^j_k|_{\mathtt{BV}} \longrightarrow |\psi^j|_{\mathtt{BV}}$ in Item c) in Definition 1 is in the sense of variation measure [21, Chapter 5]. The incorporation of item c) in the Definition 1 implies the existence of the Γ-limit of sequences of admissible quadruples [22, 1]. This appears in the proof of Lemmas 4, 5 and 8, where we proove that the set of admissible quadruples are closed in the defined topology (see Lemmas 4 and 5) and in the weak lower semi-continuity of the regularization functional R (see Lemma 8). The identification of non-constant level values ψ^j imply in a different definition of admissible quadruples.

As a consequence, the arguments in the proof of regularization properties of the level set approach are the principal theoretical novelty and the difference between our definition of admissible quadruples and the ones in [13, 14, 22].

Remark 3. *For $j = 1, 2$ let $\psi^j \in \mathbb{B} \cap C^\infty(\Omega)$, $\phi \in H^1(\Omega)$ be such that $|\nabla\phi| \neq 0$ in the neighborhood of the level set $\{\phi(x) = 0\}$ and $H(\phi) = z \in \mathcal{V}$. For each $k \in \mathbb{N}$ set $\psi^j_k = \psi^j$ and $\phi_k = \phi$. Then, for all sequences of $\{\varepsilon_k\}_{k\in\mathbb{N}}$ of positive numbers converging to zero, we have*

$$\|H_{\varepsilon_k}(\phi_k) - z\|_{L^1(\Omega)} = \|H_{\varepsilon_k}(\phi_k) - H(\phi)\|_{L^1(\Omega)} = \int_{(\phi)^{-1}[-\varepsilon_k,0]} \left|1 - \frac{\phi}{\varepsilon_k}\right| dx$$
$$\leq \int_{-\varepsilon_k}^{0} \int_{(\phi)^{-1}(\tau)} 1 d\tau \leq meas\{(\phi)^{-1}(\tau)\} \int_{-\varepsilon_k}^{0} 1 dt \longrightarrow 0\,.$$

Here, we use the fact that $|\nabla\phi| \neq 0$ in the neighborhood of $\{\phi = 0\}$ implies that ϕ is a local diffeomorphism together with a co-area formula [21, Chapter 4]. Moreover, $\{\psi^j_k\}_{k\in\mathbb{N}}$ in $\mathbb{B} \cap C^\infty(\Omega)$ satisfyes Definition 1, item c).

Hence, $(z, \phi, \psi^1, \psi^2)$ is an admissible quadruple. In particular, we conclude from the general assumption above that the set of admissible quadruple satisfying $F(u) = y$ is not empty.

3.1 Relevant Properties of Admissible Quadruples

Our first result is the proof of the continuity properties of operators P_ε, H_ε and q in suitable topologies. Such result will be necessary in the subsequent analysis.

We start with an auxiliary lemma that is well known (see for example [16]). We present it here for the sake of completeness.

Lemma 2. *Let Ω be a measurable subset of $\mathbb{R}^n$ with finite measure.*

If $(f_k) \in \mathbb{B}$ is a convergent sequence in $L^p(\Omega)$ for some p, $1 \leq p < \infty$, then it is a convergent sequence in $L^p(\Omega)$ for all $1 \leq p < \infty$.

In particular Lemma 2 holds for the sequence $z_k := H_\varepsilon(\phi_k)$.

Proof. See [16, Lemma 2.1]. □

The next two lemmas are auxiliary results in order to understand the definition of the set of admissible quadruples.

Lemma 3. *Let Ω as in assumption **(A1)** and $j = 1, 2$.*

- *(i) Let $\{z_k\}_{k\in\mathbb{N}}$ be a sequence in $L^\infty(\Omega)$ with $z_k \in [m, M]$ a.e. converging in the $L^1(\Omega)$-norm to some element z and $\{\psi^j_k\}_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}$ converging in the* BV*-norm to some $\psi^j \in \mathbb{B}$. Then $q(z_k, \psi^1_k, \psi^2_k)$ converges to $q(z, \psi^1, \psi^2)$ in $L^1(\Omega)$.*

- *(ii) Let $(z, \phi) \in L^1(\Omega) \times H^1(\Omega)$, be such that $H_\varepsilon(\phi) \to z$ in $L^1(\Omega)$ as $\varepsilon \to 0$ and let $\psi^1, \psi^2 \in \mathbb{B}$. Then $P_\varepsilon(\phi, \psi^1, \psi^2) \to q(z, \psi^1, \psi^2)$ in $L^1(\Omega)$ as $\varepsilon \to 0$.*

(iii) *Given $\varepsilon > 0$, let $\{\phi_k\}_{k\in\mathbb{N}}$ be a sequence in $H^1(\Omega)$ converging to $\phi \in H^1(\Omega)$ in the L^2-norm. Then $H_\varepsilon(\phi_k) \to H_\varepsilon(\phi)$ in $L^1(\Omega)$, as $k \to \infty$. Moreover, if $\{\psi_k^j\}_{k\in\mathbb{N}}$ are sequences in $\mathbb{B}$, converging to some ψ^j in $\mathbb{B}$, with respect to the $L^1(\Omega)$-norm, then $q(H_\varepsilon(\phi_k), \psi_k^1, \psi_k^2) \to q(H_\varepsilon(\phi), \psi^1, \psi^2)$ in $L^1(\Omega)$, as $k \to \infty$.*

Proof. Since Ω is assumed to be bounded, we have $L^\infty(\Omega) \subset L^1(\Omega)$ and $BV(\Omega)$ is continuous embedding in $L^2(\Omega)$ [21]. To prove *(i)*, notice that

$$\begin{aligned}\|q(z_k, \psi_k^1, \psi_k^2) - q(z, \psi^1, \psi^2)\|_{L^1(\Omega)} &= \left\|\psi_k^1 z_k + \psi_k^2(1 - z_k) - \psi^1 z - \psi^2(1 - z)\right\|_{L^1(\Omega)} \\ &\le \|z_k\|_{L^\infty(\Omega)}\left\|\psi_k^1 - \psi^1\right\|_{L^1(\Omega)} + \left\|\psi^1\right\|_{L^2(\Omega)}\|z_k - z\|_{L^2(\Omega)} \\ &+ \|1 - z_k\|_{L^\infty(\Omega)}\left\|\psi_k^2 - \psi^2\right\|_{L^1(\Omega)} + \left\|\psi^2\right\|_{L^2(\Omega)}\|z_k - z\|_{L^2(\Omega)} \stackrel{k\to\infty}{\longrightarrow} 0\,.\end{aligned}$$

Here we use Lemma 2 in order to guarantee the convergence of z_k to z in $L^2(\Omega)$.

Assertion *(ii)* follows with similar arguments and the fact that $H_\varepsilon(\phi) \in L^\infty(\Omega)$ for all $\varepsilon > 0$.

As $\|H_\varepsilon(\phi_k) - H_\varepsilon(\phi)\|_{L^1(\Omega)} \le \varepsilon^{-1}\sqrt{\mathrm{meas}(\Omega)}\|\phi_k - \phi\|_{L^2(\Omega)}$ the first part of assertion *(iii)* follows. The second part of the assertion *(iii)* holds by a combination of the inequality above and inequalities in the proof of assertion *(i)*. □

Lemma 4. *Let $\{\psi_k^j\}_{k\in\mathbb{N}}$ be a sequence of functions satisfying Definition 1 converging in $L^1(\Omega)$ to some ψ^j, for $j = 1, 2$. Then ψ^j also satisfies Definition 1.*

Sketch of the proof.

Let $k \in \mathbb{N}$ and $j = 1, 2$. Since ψ_k^j satisfies Definition 1, $\psi_k^j \in \mathtt{BV}$. From [21, Theorem 2, pg 172] there exist sequences $\{\psi_{k,l}^j\}_{l\in\mathbb{N}}$ in $\mathtt{BV} \times C^\infty(\Omega)$ such that

$$\psi_{k,l}^j \stackrel{l\to\infty}{\longrightarrow} \psi_k^j \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad |\psi_{k,l}^j|_{\mathtt{BV}} \stackrel{l\to\infty}{\longrightarrow} |\psi_k^j|_{\mathtt{BV}}\,.$$

In particular, for the subsequence $\{\psi_{k,l(k)}^j\}_{k\in\mathbb{N}}$ follows that

$$\psi_{k,l(k)}^j \stackrel{k\to\infty}{\longrightarrow} \psi^j \quad \text{in} \quad L^1(\Omega) \quad \text{and} \quad |\psi_{k,l(k)}^j|_{\mathtt{BV}} \stackrel{k\to\infty}{\longrightarrow} |\psi^j|_{\mathtt{BV}}\,. \tag{10}$$

Moreover, by assumption $\psi^j \in L^1(\Omega)$. From the lower semi-continuity of variational measure (see [21, Theorem 1 pg. 172]), equation (10) and the definition of BV space, it follows that $\psi^j \in \mathtt{BV}$. □

In the next lemma we prove that the set of admissible quadruples is closed with respect the $L^1(\Omega) \times L^2(\Omega) \times (L^1(\Omega))^2$ topology.

Lemma 5. *Let $(z_k, \phi_k, \psi_k^1, \psi_k^2)$ be a sequence of admissible quadruples converging in $L^1(\Omega) \times L^2(\Omega) \times (L^1(\Omega))^2$ to some $(z, \phi, \psi^1, \psi^2)$, with $\phi \in H^1(\Omega)$. Then, $(z, \phi, \psi^1, \psi^2)$ is also an admissible quadruple.*

Sketch of the proof. Let $k \in \mathbb{N}$. Since $(z_k^1, \phi_k^1, \psi_k^1, \psi_k^2)$ is an admissible quadruple, it follows from Definition 1 that there exist sequences $\{\phi_{k,l}\}_{l\in\mathbb{N}}$, in $H^1(\Omega)$, $\{\psi_{k,l}^1\}_{l\in\mathbb{N}}$, $\{\psi_{k,l}^2\}_{l\in\mathbb{N}}$ in $\mathtt{BV} \times C^\infty(\Omega)$ and a correspondent sequence $\{\varepsilon_k^l\}_{l\in\mathbb{N}}$ converging to zero such that

$$\phi_{k,l} \stackrel{l\to\infty}{\longrightarrow} \phi_k \ \text{ in } \ L^2(\Omega)\,, \quad H_{\varepsilon_k^l}(\phi_{k,l}) \stackrel{l\to\infty}{\longrightarrow} z_k \ \text{ in } \ L^1(\Omega) \quad \text{ and } \quad |\psi_{k,l}^j|_{\mathtt{BV}} \stackrel{l\to\infty}{\longrightarrow} |\psi_k^j|_{\mathtt{BV}}\,, j = 1, 2\,.$$

Define the monotone increasing function $\tau : \mathbb{N} \to \mathbb{N}$ such that, for every $k \in \mathbb{N}$ it holds

$$\varepsilon_k^{\tau(k)} \le \frac{1}{2}\varepsilon_{k-1}^{\tau(k-1)}\,,\ \left\|\phi_{k,\tau(k)} - \phi_k\right\|_{L^2(\Omega)} \le \frac{1}{k}\,,\ \left\|H_{\varepsilon_k^{\tau(k)}}(\phi_{k,\tau(k)}) - z_k\right\|_{L^1(\Omega)} \le \frac{1}{k}\,,\ |\psi_{k,\tau(k)}^j|_{\mathtt{BV}} \longrightarrow |\psi_k^j|_{\mathtt{BV}}\,, j = 1, 2\,. \tag{11}$$

Hence, for each $k \in \mathbb{N}$

$$
\begin{aligned}
\left\|\phi-\phi_{k,\tau(k)}\right\|_{L^2(\Omega)} &\leq \|\phi-\phi_k\|_{L^2(\Omega)}+\left\|\phi_{k,\tau(k)}-\phi_k\right\|_{L^2(\Omega)} \\
\left\|z-H_{\varepsilon_k^{\tau(k)}}(\phi_{k,\tau(k)})\right\|_{L^1(\Omega)} &\leq \|z-z_k\|_{L^1(\Omega)}+\left\|H_{\varepsilon_k^{\tau(k)}}(\phi_{k,\tau(k)})-z_k\right\|_{L^1(\Omega)}.
\end{aligned}
$$

From (11),

$$
\lim_{k\to\infty}\left\|\phi-\phi_{k,\tau(k)}\right\|_{L^2(\Omega)}=0\ ,\ \lim_{k\to\infty}\left\|z-H_{\varepsilon_k^{\tau(k)}}(\phi_{k,\tau(k)})\right\|_{L^1(\Omega)}=0\,. \tag{12}
$$

Moreover, with the same arguments as Lemma 4, it follows that

$$
|\psi^j_{k,\tau(k)}|_{\mathtt{BV}} \to |\psi^j|_{\mathtt{BV}}\,, \qquad j=1,2\,,
$$

and $\psi^j \in \mathtt{BV}(\Omega)$. Therefore, it remains to prove that (z,ϕ,ψ^1,ψ^2) is an admissible quadruple. From Definition 1 and Lemma 4, it is enough to prove that $z \in L^\infty(\Omega)$. If this is not the case, there would exist a $\Omega' \subset \Omega$ with $|\Omega'|>0$ and $\gamma>0$ such that $z(x)>1+\gamma$ in Ω' (the other case: $z(x)<-\gamma$ is analogous). Since $(H_{\varepsilon_k^{\tau(k)}}(\phi_{k,\tau(k)}))(x) \in [0,1]$ a.e. in Ω for $k \in \mathbb{N}$ (see remark after Definition 1), we would have

$$
\|z-H_{\varepsilon_k^{\tau(k)}}(\phi_{k,\tau(k)})\|_{L^1(\Omega)} \geq \|z-H_{\varepsilon_k^{\tau(k)}}(\phi_{k,\tau(k)})\|_{L^1(\Omega')} \geq \gamma|\Omega'|\,,\ k\in\mathbb{N}\,,
$$

contradicting the second limit in (12). ☐

3.2 Relevant Properties of the Penalization Functional

In next lemmas, we verify properties of the functional R which are fundamental for the convergence analysis outlined in Section 4. In particular, these properties implies that the level sets of $\mathcal{G}_\alpha$ are compact in the set of admissible quadruple, i.e., $\mathcal{G}_\alpha$ assume a minimizer on this set. First, we prove a lemma that simplify the functional R in (8). Here we present the sketch of the proof. For more details, see the arguments in [14, Lemma 3].

Lemma 6. *Let (z,ϕ,ψ^1,ψ^2) be an admissible quadruple. Then, there exists sequences $\{\varepsilon_k\}_{k\in\mathbb{N}}$, $\{\phi_k\}_{k\in\mathbb{N}}$ and $\{\psi^j_k\}_{k\in\mathbb{N}}$ as in the Definition 1, such that*

$$
R(z,\phi,\psi^1,\psi^2)=\lim_{k\to\infty}\left\{\beta_1|H_{\varepsilon_k}(\phi_k)|_{\mathtt{BV}}+\beta_2\|\phi_k-\phi_0\|^2_{H^1(\Omega)}+\beta_3\sum_{j=1}^{2}|\psi^j_k-\psi^j_0|_{\mathtt{BV}}\right\}. \tag{13}
$$

Sketch of the proof. For each $l \in \mathbb{N}$, the definition of R (see Definition 1) guaranties the existence of sequences ε^l_k, $\{\phi^j_{k,l}\} \in H^1(\Omega)$ and $\{\psi^j_{k,l}\} \in \mathbb{B}$ such that

$$
R(z,\phi,\psi^1,\psi^2)=\lim_{l\to\infty}\left\{\liminf_{k\to\infty}\left\{\beta_1|H_{\varepsilon^l_k}(\phi_{k,l})|_{\mathtt{BV}}+\beta_2\|\phi_{k,l}-\phi_0\|^2_{H^1(\Omega)}\right\}+\beta_3\sum_{j=1}^{2}|\psi^j_{k,l}-\psi^j_0|_{\mathtt{BV}}\right\}.
$$

Now a similar extraction of subsequences as in Lemma 5 complete the proof. ☐

In the following, we prove two lemmas that are essential to the proof of well posedness of the Tikhonov functional (5).

Lemma 7. *The functional R in (8) is coercive on the set of admissible quadruples. In other words, given any admissible quadruple (z,ϕ,ψ^1,ψ^2) we have*

$$
R(z,\phi,\psi^1,\psi^2) \geq \left(\beta_1|z|_{\mathtt{BV}}+\beta_2\|\phi-\phi_0\|^2_{H^1(\Omega)}+\beta_2\sum_{j=1}^{2}|\psi^j-\psi^j_0|_{\mathtt{BV}}\right).
$$

Sketch of the proof. Let $(z, \phi, \psi^1, \psi^2)$ be an admissible quadruple. From [15, Lemma 4], it follows that

$$\rho(z, \phi) \;\geq\; \left(\beta_1 |z|_{\mathtt{BV}} + \beta_2 \|\phi - \phi_0\|^2_{H^1(\Omega)}\right) . \tag{14}$$

Now, from (14) and the definition of R in (8), we have

$$\left(\beta_1 |z|_{\mathtt{BV}} + \beta_2 \|\phi - \phi_0\|^2_{H^1(\Omega)} + \beta_3 \sum_{j=1}^{2} |\psi^j - \psi^j_0|_{\mathtt{BV}}\right) \;\leq\; \rho(z, \phi) + \beta_3 \sum_{j=1}^{2} |\psi^j - \psi^j_0|_{\mathtt{BV}} \;=\; R(z, \phi, \psi^1, \psi^2) \,,$$

concluding the proof. □

Lemma 8. *The functional R in (8) is weak lower semi-continuous on the set of admissible quadruples, i.e. given a sequence $\{(z_k, \phi_k, \psi^1_k, \psi^2_k)\}$ of admissible quadruples such that $z_k \to z$ in $L^1(\Omega)$, $\phi_k \rightharpoonup \phi$ in $H^1(\Omega)$, $\psi^j_k \to \psi^j$ in $L^1(\Omega)$, for some admissible quadruple $(z, \phi, \psi^1, \psi^2)$, then*

$$R(z, \phi, \psi^1, \psi^2) \;\leq\; \liminf_{k \in \mathbb{N}} R(z_k, \phi_k, \psi^1_k, \psi^2_k) \,.$$

Proof. The functional $\rho(z, \phi)$ is weak lower semi-continuous cf. [15, Lemma 5]. As $\psi^j_k \in \mathtt{BV}$ follows from [21, Theorem 2 pg 172] that there exist sequences $\{\psi^j_{k,l}\} \in \mathtt{BV} \cap C^\infty(\Omega)$ such that $\left\|\psi^j_{k,l} - \psi^j_k\right\|_{L^1(\Omega)} \leq \frac{1}{l}$. From a diagonal argument, we can extract a subsequence $\{\psi^j_{k,l(k)}\}$ of $\{\psi^j_{k,l}\}$ such that $\{\psi^j_{k,l(k)}\} \to \psi^j$ in $L^1(\Omega)$ as $k \to \infty$. Let $\xi \in C^1_c(\Omega, \mathbb{R}^n)$, $|\xi| \leq 1$. Then, from [21, Theorem 1 pg 167], it follows that

$$\begin{aligned}\int_\Omega \psi^j \, \nabla \cdot \xi dx = \lim_{k \to \infty} \int_\Omega \psi^j_{k,l(k)} \, \nabla \cdot \xi dx = \lim_{k \to \infty} \left[\int_\Omega \left(\psi^j_{k,l(k)} - \psi^j_k\right) \, \nabla \cdot \xi dx + \int_\Omega \psi^j_k \, \nabla \cdot \xi dx\right] \\ \leq \lim_{k \to \infty} \left[\left\|\psi^j_{k,l(k)} - \psi^j_k\right\|_{L^1(\Omega)} \|\nabla \cdot \xi\|_{L^\infty(\Omega)} |\Omega| - \int_\Omega \xi \cdot \sigma_k d|\psi^j_k|_{\mathtt{BV}}\right] \leq \liminf_{k \to \infty} |\psi^j_k|_{\mathtt{BV}} \,.\end{aligned}$$

Thus, form the definition of $|\cdot|_{\mathtt{BV}}$ (see [21]), we have

$$|\psi^j|_{\mathtt{BV}} = \sup\left\{\int_\Omega \psi^j \, \nabla \cdot \xi dx \,; \xi \in C^1_c(\Omega, \mathbb{R}^n) \,, \, |\xi| \leq 1\right\} \leq \liminf_{k \to \infty} |\psi^j_k|_{\mathtt{BV}} \,.$$

Now, the lemma follows from the fact that the functional R in (8) is a linear combination of lower semi-continuous functionals. □

4 Convergence Analysis

In the following, we consider any positive parameter α, β_j, $j = 1, 2, 3$ as in the general assumption to this article. First, we prove that the functional $\mathcal{G}_\alpha$ in (7) is well posed.

Theorem 9 (Well-Posedness)**.** *The functional $\mathcal{G}_\alpha$ in (7) attains minimizers on the set of admissible quadruples.*

Proof. Notice that, the set of admissible quadruples is not empty, since $(0, 0, 0, 0)$ is admissible. Let $\{(z_k, \phi_k, \psi^1_k, \psi^2_k)\}$ be a minimizing sequence for $\mathcal{G}_\alpha$, i.e. a sequence of admissible quadruples satisfying $\mathcal{G}_\alpha(z_k, \phi_k, \psi^1_k, \psi^2_k) \to \inf \mathcal{G}_\alpha \leq \mathcal{G}_\alpha(0, 0, 0, 0) < \infty$. Then, $\{\mathcal{G}_\alpha(z_k, \phi_k, \psi^1_k, \psi^2_k)\}$ is a bounded sequence of real numbers. Therefore, $\{(z_k, \phi_k, \psi_k^1, \psi_k^2)\}$ is uniformly bounded in $\mathtt{BV} \times H^1(\Omega) \times \mathtt{BV}^2$. Thus, from the Sobolev Embedding Theorem [2, 21], we guarantee the existence of a subsequence (denoted again by $\{(z_k, \phi_k, \psi_k^1, \psi_k^2)\}$) and the existence of $(z, \phi, \psi^1, \psi^2) \in L^1(\Omega) \times H^1(\Omega) \times \mathtt{BV}^2$ such that $\phi_k \to \phi$ in $L^2(\Omega)$, $\phi_k \rightharpoonup \phi$ in $H^1(\Omega)$, $z_k \to z$ in $L^1(\Omega)$ and $\psi_k^j \to \psi^j$ in $L^1(\Omega)$. Moreover, z, ψ^1 and $\psi^2 \in \mathtt{BV}$. See [21, Theorem 4, pp. 176].

From Lemma 5, we conclude that $(z, \phi, \psi^1, \psi^2)$ is an admissible quadruple. Moreover, from the weak lower semi-continuity of R (Lemma 8), together with the continuity of q (Lemma 3) and continuity of F (see the general assumption), we obtain

$$\begin{aligned} \inf\ \mathcal{G}_\alpha &= \lim_{k\to\infty} \mathcal{G}_\alpha(z_k, \phi_k, \psi_k^1, \psi_k^2) = \lim_{k\to\infty} \left\{ \|F(q(z_k, \psi_k^1, \psi_k^2)) - y^\delta\|_Y^2 + \alpha R(z_k, \phi_k, \psi_k^1, \psi_k^2) \right\} \\ &\geq \|F(q(z, \psi^1, \psi^2)) - y^\delta\|_Y^2 + \alpha R(z, \phi, \psi^1, \psi^2) = \mathcal{G}_\alpha(z, \phi, \psi^1, \psi^2)\,, \end{aligned} \tag{15}$$

proving that $(z, \phi, \psi^1, \psi^2)$ minimizes $\mathcal{G}_\alpha$. ☐

In that follows, we shall denote a minimizer of $\mathcal{G}_\alpha$ by $(z_\alpha, \phi_\alpha, \psi_\alpha^1, \psi_\alpha^2)$. In particular the functional $\hat{\mathcal{G}_\alpha}$ in (22) attain a generalized minimizer in the sense of Definition 1. In the next theorem, we summarize some convergence results for the regularized minimizers. These results are based on the existence of a generalized *minimum norm solutions*.

Definition 2. *An admissible quadruple $(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})$ is called a R-minimizing solution if satisfies*

$$\begin{aligned} &(i)\ F(q(z^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})) = y\,, \\ &(ii)\ R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger}) = ms := \inf \left\{ R(z, \phi, \psi^1, \psi^2)\,;\ (z, \phi, \psi^1, \psi^2) \text{ is an} \right. \\ &\qquad\qquad \left. \text{admissible quadruple and } F(q(z, \psi^1, \psi^2)) = y \right\}\,. \end{aligned}$$

Theorem 10 (R-minimizing solutions)**.** *Under the general assumptions of this paper there exists a R-minimizing solution.*

Proof. From the general assumption on this paper and Remark 3, we conclude that the set of admissible quadruple satisfying $F(q(z, \psi^1, \psi^2)) = y$ is not empty. Thus, ms in (ii) is finite and there exists a sequence $\{(z_k, \phi_k, \psi_k^1, \psi_k^2)\}_{k\in\mathbb{N}}$ of admissible quadruple satisfying

$$F(q(z_k, \psi_k^1, \psi_k^2)) = y \quad \text{and} \quad R(z_k, \phi_k, \psi_k^1, \psi_k^2) \to ms < \infty\,.$$

Now, form the definition of R, it follows that the sequences $\{\phi_k\}_{k\in\mathbb{N}}$, $\{z_k\}_{k\in\mathbb{N}}$ and $\{\psi_k^j\}_{k\in\mathbb{N}}^{j=1,2}$ are uniformly bounded in $H^1(\Omega)$ and $\mathtt{BV}(\Omega)$, respectively. Then, from the Sobolev Compact Embedding Theorem [2, 21], we have (up to subsequences) that

$$\phi_k \to \phi^\dagger \ \text{ in } \ L^2(\Omega)\,, \quad z_k \to z^\dagger \ \text{ in } \ L^1(\Omega) \quad \text{ and } \quad \psi_k^j \to \psi^{j,\dagger} \ \text{ in } \ L^1(\Omega)\,,\ j = 1, 2\,.$$

Lemma 5 implies that $(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})$ is an admissible quadruple. Since R is weakly lower semi-continuous (cf. Lemma 8), it follows

$$ms = \liminf_{k\to\infty} R(z_k, \phi_k, \psi_k^1, \psi_k^2) \geq R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})\,.$$

Moreover, we conclude from Lemma 3 that

$$q(z^\dagger, \psi^{1,\dagger}, \psi^{1,\dagger}) = \lim_{k\to\infty} q(z_k, \psi_k^1, \psi_k^2) \quad \text{ and } \quad F(q(z^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})) = \lim_{k\to\infty} F(q(z_k, \psi_k^1, \psi_k^2)) = y\,.$$

Thus, $(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})$ is a R- minimizing solution. ☐

Using classical techniques from the analysis of Tikhonov regularization methods (see [20, 19]), we present below the main convergence and stability theorems of this paper. The arguments in the proof are somewhat different of that presented in [14, 13]. But, for sake of completeness, we present the proof.

Theorem 11 (**Convergence for exact data).** *Assume that we have exact data, i.e.* $y^\delta = y$. *For every* $\alpha > 0$ *let* $(z_\alpha, \phi_\alpha, \psi^1_\alpha, \psi^2_\alpha)$ *denote a minimizer of* $\mathcal{G}_\alpha$ *on the set of admissible quadruples. Then, for every sequence of positive numbers* $\{\alpha_k\}_{k\in\mathbb{N}}$ *converging to zero there exists a subsequence, denoted again by* $\{\alpha_k\}_{l\in\mathbb{N}}$, *such that* $(z_{\alpha_k}, \phi_{\alpha_k}, \psi^1_{\alpha_k}, \psi^2_{\alpha_k})$ *is strongly convergent in* $L^1(\Omega) \times L^2(\Omega) \times (L^1(\Omega))^2$. *Moreover, the limit is a solution of* (1).

Proof. Let $(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})$ be a R-minimizing solution of (1) – its existence is guaranteed by Theorem 10. Let $\{\alpha_k\}_{k\in\mathbb{N}}$ be a sequence of positive numbers converging to zero. For each $k \in \mathbb{N}$, denote $(z_k, \phi_k, \psi^1_k, \psi^2_k) := (z_{\alpha_k}, \phi_{\alpha_k}, \psi^1_{\alpha_k}, \psi^2_{\alpha_k})$ be a minimizer of G_{α_k}. Then, for each $k \in \mathbb{N}$, we have

$$G_{\alpha_k}(z_k, \phi_k, \psi^1_k, \psi^2_k) \le \left\| F(q(z^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})) - y \right\| + \alpha_k R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger}) = \alpha_k R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger}). \tag{16}$$

Since $\alpha_k R(z_k, , \phi_k, \psi^1_k, \psi^2_k) \le G_{\alpha_k}(z_k, \phi_k, \psi^1_k, \psi^2_k)$, it follows from (16) that

$$R(z_k, \phi_k, \psi^1_k, \psi^2_k) \ \le \ R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger}) \ < \ \infty \, . \tag{17}$$

Moreover, from the assumption on the sequence $\{\alpha_k\}$, it follows that

$$\lim_{k\to\infty} \ \alpha_k R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{1,\dagger}) \ = \ 0 \, . \tag{18}$$

From (17) and Lemma 7, we conclude that sequences $\{\phi_k\}$, $\{z_k\}$ and $\{\psi^j_k\}$ are bounded in $H^1(\Omega)$ and $\mathtt{BV}$, respectively, for $j = 1, 2$. Using an argument of extraction of diagonal subsequences (see proof of Lemma 5), we can guarantee the existence of an admissible quadruple $(\tilde{z}, \tilde{\phi}, \tilde{\psi}^1, \tilde{\psi}^2)$ such that

$$(z_k, \phi_k, \psi^1_k, \psi^2_k) \to (\tilde{z}, \tilde{\phi}, \tilde{\psi}^1, \tilde{\psi}^2) \ \text{ in } \ L^1(\Omega) \times L^2(\Omega) \times (L^1(\Omega))^2 \, .$$

Now, from Lemma 3 (i), it follows that $q(\tilde{z}, \tilde{\psi}^1, \tilde{\psi}^2) = \lim\limits_{k\to\infty} q(z_k, \psi^1_k, \psi^2_k)$ in $L^1(\Omega)$. Using the continuity of the operator F together with (16) and (18), we conclude that

$$y \ = \ \lim_{k\to\infty} F(q(z_k, \psi^1_k, \psi^2_k)) \ = \ F(q(\tilde{z}, \tilde{\psi}^1, \tilde{\psi}^2)) \, .$$

On the other hand, from the lower semi-continuity of R and (17) it follows that

$$R(\tilde{z}, \tilde{\phi}, \tilde{\psi}^1, \tilde{\psi}^2) \ \le \ \liminf_{k\to\infty} R(z_k, \phi_k, \psi^1_k, \psi^2_k) \ \le \ \limsup_{k\to\infty} R(z_k, \phi_k, \psi^1_k, \psi^2_k)) \ \le \ R(z^\dagger, \phi^\dagger, \tilde{\psi}^1, \tilde{\psi}^2) \, ,$$

concluding the proof. □

Theorem 12 (**Stability).** *Let* $\alpha = \alpha(\delta)$ *be a function satisfying* $\lim\limits_{\delta\to 0} \alpha(\delta) = 0$ *and* $\lim\limits_{\delta\to 0} \delta^2 \alpha(\delta)^{-1} = 0$. *Moreover, let* $\{\delta_k\}_{k\in\mathbb{N}}$ *be a sequence of positive numbers converging to zero and* $y^{\delta_k} \in Y$ *be corresponding noisy data satisfying* (2). *Then, there exist a subsequence, denoted again by* $\{\delta_k\}$, *and a sequence* $\{\alpha_k := \alpha(\delta_k)\}$ *such that* $(z_{\alpha_k}, \phi_{\alpha_k}, \psi^1_{\alpha_k}, \psi^2_{\alpha_k})$ *converges in* $L^1(\Omega) \times L^2(\Omega) \times (L^1(\Omega))^2$ *to solution of* (1).

Proof. Let $(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{1,\dagger})$ be a R-minimizer solution of (1) (such existence is guaranteed by Theorem 10). For each $k \in \mathbb{N}$, let $(z_k, \phi_k, \psi_k^1, \psi_k^2) := (z_{\alpha(\delta_k)}, \phi_{\alpha(\delta_k)}, \psi^1_{\alpha(\delta_k)}, \psi^2_{\alpha(\delta_k)})$ be a minimizer of $G_{\alpha(\delta_k)}$. Then, for each $k \in \mathbb{N}$ we have

$$\begin{aligned} G_{\alpha_k}(z_k, \phi_k, \psi_k^1, \psi_k^2) &\le \left\| F(q(z^\dagger, \psi^{1,\dagger}, \psi^{1,\dagger})) - y^{\delta_k} \right\|_Y^2 + \alpha(\delta_k) R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger}) \\ &\le \ \delta_k^2 + \alpha(\delta_k) R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})\,. \end{aligned} \tag{19}$$

From (19) and the definition of G_{α_k}, it follows that

$$R(z_k, \phi_k, \psi_k^1, \psi_k^2) \ \le \ \frac{\delta_k^2}{\alpha(\delta_k)} + R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})\ . \tag{20}$$

Taking the limit as $k \to \infty$ in (20), it follows from theorem assumptions on $\alpha(\delta_k)$, that

$$\lim_{k\to\infty} \left\| F(q(z_k, \psi_k^1, \psi_k^2)) - y^{\delta_k} \right\| \ \le \ \lim_{k\to\infty} \left(\delta_k^2 + \alpha(\delta_k) R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger}) \right) \ = \ 0\,,$$

and

$$\limsup_{k\to\infty} R(z_k, \phi_k, \psi_k^1, \psi_k^2) \ \le \ R(z^\dagger, \phi^\dagger, \psi^{1,\dagger}, \psi^{2,\dagger})\ . \tag{21}$$

With the same arguments as in the proof of Theorem 11, we conclude that, at least a subsequence that we denote again by $(z_k, \phi_k, \psi_k^1, \psi_k^2)$, converge in $L^1(\Omega) \times L^2(\Omega) \times (L^1(\Omega))^2$ to some admissible quadruple $(z, \phi, \psi^1, \psi^2)$. Moreover, by taking the limit as $k \to \infty$ in (19), it follows from the assumption on F and Lemma 3 that

$$F(q(z, \phi, \psi^1, \psi^2)) = \lim_{k\to\infty} F(q(z_k, \psi_k^1, \psi_k^2)) = y\,.$$

$\square$

The functional $\mathcal{G}_\alpha$ defined in (7) is not easy to handled numerically, i.e., we are not able to derive a suitable optimality condition to the minimizers of $\mathcal{G}_\alpha$. In the next section, we work in sight to surpass such difficulty.

5 Numerical Solution

In this section, we introduce a functional which can be handled numerically, and whose minimizers are 'near' to the minimizers of $\mathcal{G}_\alpha$. Let $\mathcal{G}_{\varepsilon,\alpha}$ be the functional defined by

$$\mathcal{G}_{\varepsilon,\alpha}(\phi, \psi^1, \psi^2) := \|F(P_\varepsilon(\phi, \psi^1, \psi^2)) - y^\delta\|_Y^2 + \alpha\big(\beta_1 |H_\varepsilon(\phi)|_{\mathtt{BV}} + \beta_2 \|\phi - \phi_0\|_{H^1}^2 + \beta_3 \sum_{j=1}^2 |\psi^j - \psi_0^j|_{\mathtt{BV}}\big), \tag{22}$$

where $P_\varepsilon(\phi, \psi^1, \psi^2) := q(H_\varepsilon(\phi), \psi^1, \psi^2)$ is defined in (6). The functional $\mathcal{G}_{\varepsilon,\alpha}$ is well-posed as the following lemma shows:

Lemma 13. *Given positive constants α, ε, β_j as in the general assumption of this article, $\phi_0 \in H^1(\Omega)$ and $\psi_0^j \in \mathbb{B}$, $j = 1, 2$. Then, the functional $\mathcal{G}_{\varepsilon,\alpha}$ in (22) attains a minimizer on $H^1(\Omega) \times (\mathtt{BV})^2$.*

Proof. Since, $\inf\{\mathcal{G}_{\varepsilon,\alpha}(\phi,\psi^1,\psi^2) \,:\, (\phi,\psi^1,\psi^2)\in H^1(\Omega)\times(\mathtt{BV})^2\} \le \mathcal{G}_{\varepsilon,\alpha}(0,0,0)<\infty$, there exists a minimizing sequence $\{(\phi_k,\psi_k^1,\psi_k^2)\}$ in $H^1(\Omega)\times\mathbb{B}^2$ satisfying

$$\lim_{k\to\infty}\mathcal{G}_{\varepsilon,\alpha}(\phi_k,\psi_k^1,\psi_k^2) \;=\; \inf\{\mathcal{G}_{\varepsilon,\alpha}(\phi,\psi^1,\psi^2) \,:\, (\phi,\psi^1,\psi^2)\in H^1(\Omega)\times\mathbb{B}^2\}\,.$$

Then, for fixed $\alpha>0$, the definition of $\mathcal{G}_{\varepsilon,\alpha}$ in (22) implies that the sequences $\{\phi_k\}$ and $\{\psi_k^j\}^{j=1,2}$ are bounded in $H^1(\Omega)$ and $(\mathtt{BV})^2$, respectively. Therefore, from Banach-Alaoglu-Bourbaki Theorem [43] $\phi_k\rightharpoonup\phi$ in $H^1(\Omega)$ and from [21, Theorem 4 pg. 176], $\psi_k^j\to\psi^j$ in $L^1(\Omega)$, $j=1,2$. Now, a similar argument as in Lemma 4 implies that $\psi^j\in\mathbb{B}$, for $j=1,2$. Moreover, by the weak lower semi-continuity of the H^1–norm [43] and $|\cdot|_{\mathtt{BV}}$ measure (see [21, Theorem 1 pg. 172]), it follows that

$$\|\phi-\phi_0\|_{H^1}^2 \le \liminf_{k\to\infty}\|\phi_k-\phi_0\|_{H^1}^2 \;\text{ and }\; |\psi^j-\psi_0^j|_{\mathtt{BV}} \le \liminf_{k\to\infty}|\psi_k^j-\psi_0^j|_{\mathtt{BV}}\,.$$

The compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$ [2] implies in the existence of a subsequence of $\{\phi_k\}$, (that we denote with the same index) such that $\phi_k\to\phi$ in $L^2(\Omega)$. Follows from Lemma 3 and [21, Theorem 1, pg 172] that $|H_\varepsilon(\phi)|_{\mathtt{BV}}\le\liminf\limits_{k\to\infty}|H_\varepsilon(\phi_k)|_{\mathtt{BV}}$. Hence, from continuity of F in L^1, continuity of q (see Lemma 3), together with the estimates above, we conclude that

$$\begin{aligned}\mathcal{G}_{\varepsilon,\alpha}(\phi,\psi^1,\psi^2) &\le \lim_{k\to\infty}\|F(P_\varepsilon(\phi_k,\psi_k^1,\psi_k^2))-y^\delta\|_Y^2\\ &\qquad+\alpha\Big(\beta_1\liminf_{k\to\infty}|H_\varepsilon(\phi_k)|_{\mathtt{BV}}+\beta_2\liminf_{k\to\infty}\|\phi_k-\phi_0\|_{H^1(\Omega)}^2+\beta_3\liminf_{k\to\infty}\sum_{j=1}^2|\psi_k^j-\psi_0^j|_{\mathtt{BV}}\Big)\\ &\le \liminf_{k\to\infty}\mathcal{G}_{\varepsilon,\alpha}(\phi_k,\psi_k^1,\psi_k^2) \;=\; \inf\mathcal{G}_{\varepsilon,\alpha}\,,\end{aligned}$$

Therefore, (ϕ,ψ^1,ψ^2) is a minimizer of $\mathcal{G}_{\varepsilon,\alpha}$. ◻

In the sequel, we prove that, when $\varepsilon\to0$, the minimizers of $\mathcal{G}_{\varepsilon,\alpha}$ approximate a minimizer of the functional $\mathcal{G}_\alpha$. Hence, numerically, the minimizer of $\mathcal{G}_{\varepsilon,\alpha}$ can be used as a suitable approximation for the minimizers of $\mathcal{G}_\alpha$.

Theorem 14. *Let α and β_j be given as in the general assumption of this article. For each $\varepsilon>0$, denote by $(\phi_{\varepsilon,\alpha},\psi_{\varepsilon,\alpha}^1,\psi_{\varepsilon,\alpha}^2)$ a minimizer of $\mathcal{G}_{\varepsilon,\alpha}$ (that there exist form Lemma 13). Then, there exists a sequence of positive numbers $\varepsilon_k\to0$ such that $(H_{\varepsilon_k}(\phi_{\varepsilon_k,\alpha}),\phi_{\varepsilon_k,\alpha},\psi_{\varepsilon_k,\alpha}^1,\psi_{\varepsilon_k,\alpha}^2)$ converges strongly in $L^1(\Omega)\times L^2(\Omega)\times(L^1(\Omega))^2$ and the limit minimizes $\mathcal{G}_\alpha$ on the set of admissible quadruples.*

Proof. Let $(z_\alpha,\phi_\alpha,\psi_\alpha^1,\psi_\alpha^2)$ be a minimizer of the functional $\mathcal{G}_\alpha$ on the set of admissible quadruples (cf. Theorem 9). From Definition 1, there exists a sequence $\{\varepsilon_k\}$ of positive numbers converging to zero and corresponding sequences $\{\phi_k\}$ in $H^1(\Omega)$ satisfying $\phi_k\to\phi_\alpha$ in $L^2(\Omega)$, $H_{\varepsilon_k}(\phi_k)\to z_\alpha$ in $L^1(\Omega)$ and, finally, sequences $\{\psi_k^j\}$ in $\mathtt{BV}\times C_c^\infty(\Omega)$ such that $|\psi_k^j|_{\mathtt{BV}}\longrightarrow|\psi^j|_{\mathtt{BV}}$. Moreover, we can further assume (see Lemma 6) that

$$R(z_\alpha,\phi_\alpha,\psi_\alpha^1,\psi_\alpha^2) \;=\; \lim_{k\to\infty}\big(\beta_1|H_{\varepsilon_k}(\phi_k)|_{\mathtt{BV}}+\beta_2\|\phi_k-\phi_0\|_{H^1(\Omega)}^2+\beta_3\sum_{j=1}^2|\psi_k^j-\psi_0^j|_{\mathtt{BV}}\big)\,.$$

Let $(\phi_{\varepsilon_k},\psi_{\varepsilon_k}^1,\psi_{\varepsilon_k}^2)$ be a minimizer of $\mathcal{G}_{\varepsilon_k,\alpha}$. Hence, $(\phi_{\varepsilon_k},\psi_{\varepsilon_k}^1,\psi_{\varepsilon_k}^2)$ belongs to $H^1(\Omega)\times\mathbb{B}^2$ (see Lemma 13). The sequences $\{H_{\varepsilon_k}(\phi_{\varepsilon_k})\}$,$\{\phi_{\varepsilon_k}\}$ and $\{\psi_{\varepsilon_k}^j\}$ are uniformly bounded in $\mathtt{BV}(\Omega)$, $H^1(\Omega)$ and $\mathtt{BV}(\Omega)$, for $j=1,2$, respectively. Form compact embedding (see Theorems [2] and [21, Theorem 4 pg. 176]), there exist convergent subsequences whose limits are denoted by $\tilde z$, $\tilde\phi$ and $\tilde\psi^j$ belong to $\mathtt{BV}(\Omega)$,$H^1(\Omega)$ and $\mathtt{BV}(\Omega)$, for $j=1,2$, respectively.

Summarizing, we have $\phi_{\varepsilon_k} \to \tilde{\phi}$ in $L^2(\Omega)$, $H_{\varepsilon_k}(\phi_{\varepsilon_k}) \to \tilde{z}$ in $L^1(\Omega)$, and $\psi^j_{\varepsilon_k} \to \tilde{\psi}^j$ in $L^1(\Omega)$, $j = 1, 2$. Thus, $(\tilde{z}, \tilde{\phi}, \tilde{\psi}^1, \tilde{\psi}^2) \in L^1(\Omega) \times H^1(\Omega) \times Ł^1(\Omega)$ is an admissible quadruple (cf. Lemma 5).

From the definition of R, Lemma 3 and the continuity of F, it follows that

$$\begin{aligned} \|F(q(\tilde{z}, \tilde{\psi}^1, \tilde{\psi}^2)) - y^\delta\|^2_Y \;&=\; \lim_{k\to\infty} \|F(P_{\varepsilon_k}(\phi_{\varepsilon_k}, \psi^1_{\varepsilon_k}, \psi^2_{\varepsilon_k})) - y^\delta\|^2_Y \,, \\ R(\tilde{z}, \tilde{\phi}, \tilde{\psi}^1, \tilde{\psi}^2) \;&\leq\; \liminf_{k\to\infty} \Big(\beta_1 |H_{\varepsilon_k}(\phi_{\varepsilon_k})|_{\mathtt{BV}} + \beta_2 \|\phi_{\varepsilon_k} - \phi_0\|^2_{H^1(\Omega)} + \beta_3 \textstyle\sum_{j=1}^2 |\psi^j_{\varepsilon_k} - \psi^j_0|_{\mathtt{BV}}\Big) \,. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathcal{G}_\alpha(\tilde{z}, \tilde{\phi}, \tilde{\psi}^1, \tilde{\psi}^2) \;&=\; \|F(q(\tilde{z}, \tilde{\psi}^1, \tilde{\psi}^2)) - y^\delta\|^2_Y + \alpha R(\tilde{z}, \tilde{\phi}, \tilde{\psi}^1, \tilde{\psi}^2) \\ &\leq\; \liminf_{k\to\infty}\; \mathcal{G}_{\varepsilon_k,\alpha}(\phi_{\varepsilon_k}, \psi^1_{\varepsilon_k}, \psi^2_{\varepsilon_k}) \;\leq\; \liminf_{k\to\infty} \mathcal{G}_{\varepsilon_k,\alpha}(\phi_k, \psi^1_k, \psi^2_k) \\ &\leq\; \limsup_{k\to\infty} \|F(P_{\varepsilon_k}(\phi_k, \psi^1_k, \psi^2_k)) - y^\delta\|^2_Y \\ &\qquad + \alpha \limsup_{k\to\infty} \big(\beta_1 |H_{\varepsilon_k}(\phi_k)|_{\mathtt{BV}} + \beta_2 \|\phi_k - \phi_0\|^2_{H^1(\Omega)} + \beta_3 \textstyle\sum_{j=1}^2 |\psi^j_k - \psi^j_0|_{\mathtt{BV}}\big) \\ &=\; \|F(q(z_\alpha, \psi^1_\alpha, \psi^2_\alpha)) - y^\delta\|^2_Y + \alpha R(z_\alpha, \phi_\alpha, \psi^1_\alpha, \psi^2_\alpha) \;=\; \mathcal{G}_\alpha(z_\alpha, \phi^1_\alpha, \psi^1_\alpha, \psi^2_\alpha) \,, \end{aligned}$$

characterizing $(\tilde{z}, \tilde{\phi}, \psi^1_\alpha, \psi^2_\alpha)$ as a minimizer of $\mathcal{G}_\alpha$. □

5.1 Optimality Conditions for the Stabilized Functional

For numerical purposes it is convenient to derive first order optimality conditions for minimizers of the functional $\mathcal{G}_\alpha$. Since P is a discontinuous operator, it is not possible. However, thanks to the Theorem 12, the minimizers of the stabilized functionals $\mathcal{G}_{\varepsilon,\alpha}$ can be used for approximate minimizers of the functional $\mathcal{G}_\alpha$. Therefore, we consider $\mathcal{G}_{\varepsilon,\alpha}$ in (22) ,with Y a Hilbert space, and we look for the Gâteaux directional derivatives with respect to ϕ and the unknown ψ^j for $j = 1, 2$.

Since $H'_\varepsilon(\phi)$ is self-adjoint[1], we can write the optimality conditions for the functional $\mathcal{G}_{\varepsilon,\alpha}$ in the form of the system

$$\alpha(\Delta - I)(\phi - \phi_0) \;=\; L_{\varepsilon,\alpha,\beta}(\phi, \psi^1, \psi^2)\,, \qquad \text{in } \Omega \tag{23a}$$

$$(\phi - \phi_0)\cdot\nu \;=\; 0\,, \qquad \text{at } \partial\Omega \tag{23b}$$

$$\alpha\, \nabla\cdot\big[\nabla(\psi^j - \psi^j_0)/|\nabla(\psi^j - \psi^j_0)|\big] \;=\; L^j_{\varepsilon,\alpha,\beta}(\phi, \psi^1, \psi^2)\,, \quad j = 1, 2\,. \tag{23c}$$

Here $\nu(x)$ represents the external unit normal quadruple at $x \in \partial\Omega$, and

$$\begin{aligned} L_{\varepsilon,\alpha,\beta}(\phi, \psi^1, \psi^2) \;&=\; (\psi^1 - \psi^2)\beta_2^{-1} H'_\varepsilon(\phi)^* F'(P_\varepsilon(\phi, \psi^1, \psi^2))^* (F(P_\varepsilon(\phi, \psi^1, \psi^2)) - y^\delta) \\ &\qquad - \beta_1 (2\beta_2)^{-1} H'_\varepsilon(\phi)\, \nabla\cdot\big[\nabla H_\varepsilon(\phi)/|\nabla H_\varepsilon(\phi)|\big]\,, \end{aligned} \tag{24a}$$

$$L^1_{\varepsilon,\alpha,\beta}(\phi, \psi^1, \psi^2) \;=\; (2\beta_3)^{-1}\big(F'(P_\varepsilon(\phi, \psi^1, \psi^2))\, H_\varepsilon(\phi)\big)^* (F(P_\varepsilon(\phi, \psi^1, \psi^2)) - y^\delta) \tag{24b}$$

$$L^2_{\varepsilon,\alpha,\beta}(\phi, \psi^1, \psi^2) \;=\; (2\beta_3)^{-1}\big(F'(P_\varepsilon(\phi, \psi^1, \psi^2))\, (1 - H_\varepsilon(\phi))\big)^* (F(P_\varepsilon(\phi, \psi^1, \psi^2)) - y^\delta)\,. \tag{24c}$$

It is worth noticing that the derivation of (23) is purely formal, since the BV seminorm is not differentiable. Moreover the terms $|\nabla H_\varepsilon(\phi)|$ and $|\nabla(\psi^j - \psi^j_0)|$ appearing in the denominators of (23) and (24), respectively.

In Section 6, system (23) and (24) is used as starting point for the derivation of a level set type method.

[1] Note that $H'_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon} & t \in (-\varepsilon, 0) \\ 0 & \text{other else}\,. \end{cases}$

6 Inverse Elliptic Problems

In this section, we discuss the proposed level set approach and their application in some physical problems model by elliptic PDE's. We also discuss briefly the numerical implementations of the iterative method based on the level set approach. We remark that, in the case of noise data the iterative algorithm derived by the level set approach need an early stooping criteria [19].

6.1 The Inverse Potential Problem

In this subsection, we apply the level set regularization framework in an inverse potential problem [40, 13, 23]. Differently from [39, 40, 13, 14, 22, 37, 38], we assume that the source u is not necessarily piecewise constant. For relevant applications of the inverse potential problem see [23, 24, 40, 39] and references therein.

The forward problem consists of solving the Poisson boundary value problem

$$-\nabla\cdot(\sigma\nabla w) \;=\; u\,,\ \text{in}\ \Omega\,,\quad \gamma_1 w+\gamma_2 w_\nu \;=\; g\ \text{on}\ \partial\Omega\,, \tag{25}$$

on a given domain $\Omega\subset\mathbb{R}^n$ with $\partial\Omega$ Lipschitz, for a given source function $u\in L^2(\Omega)$ and a boundary function $g\in L^2(\partial\Omega)$. In (25), ν represent the outer normal vector to $\partial\Omega$, σ is a known sufficient smooth function. Note that, depending of $\gamma_1,\gamma_2\in\{0,1\}$, we have Dirichlet, Neumann or Robin boundary condition. In this paper, we only consider the case of Dirichlet boundary condition, that corresponds to $\gamma_1=1$ and $\gamma_2=0$ in (25). Therefore, it is well known that there exists a unique solution $w\in H^1(\Omega)$ of (25) with $w-g\in H_0^1(\Omega)$, [12].

Assuming homogeneous Dirichlet boundary condition in (25), the problem can be modeled by the operator equation

$$\begin{aligned} F_1\;:\;L^2(\Omega)&\to L^2(\partial\Omega)\\ u&\longmapsto F_1(u):=w_\nu|_{\partial\Omega}\,. \end{aligned}\tag{26}$$

The corresponding inverse problem consists in recover the L^2 source function u, from measurements of the Cauchy data of its corresponding potential w on the boundary of Ω.

Using this notation, the inverse potential problem can be written in the abbreviated form $F_1(u)=y^\delta$, where the available noisy data $y^\delta\in L^2(\partial\Omega)$ have the same meaning as in (2). It is worth noticing that this inverse problem has, in general, non unique solution [23]. Therefore, we restrict our attention to minimum-norm solutions [19]. Sufficient conditions for identifiability are given in [24]. Moreover, we restrict our attention to solve the inverse problem (26) in $D(F)$, i.e., we assume that the unknown parameter $u\in D(F)$, as defined in Section 3. Note that, in this situation, the operator F_1 is linear. However, the inverse potential problem is well known to be exponentially ill-posed [24]. Therefore, the solution call for a regularization strategy [19, 23, 24].

The following lemma implies that the operator F_1 satisfies the Assumption **(A2)**.

Lemma 15. *The operator $F_1\,:\,D(F)\subset L^1(\Omega)\longrightarrow L^2(\partial\Omega)$ is continuous with the respect to the $L^1(\Omega)$ topology.*

Proof. It is well known form the elliptic regularity theory [12] that $\|w\|_{H^1(\Omega)}\le c_1\|u\|_{L^2(\Omega)}$. Let $u_n,u_0\in D(F)$ and w_n,w_0 the respective solution of (25). Then, the linearity and continuity of the trace operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ [12], we have that

$$\|F_1(u_n)-F_1(u_0)\|_{L^2(\partial\Omega)}\le C\|w_n-w_0\|_{H^1(\Omega)}\le\tilde{C}\|u_n-u_0\|_{L^2(\Omega)}\le\tilde{C}_1\|u_n-u_0\|_{L^1(\Omega)}\,,$$

where we use Lemma 2 to obtain the last inequality. Therefore, F_1 is sequentially continuous on the $L^1(\Omega)$ topology. Since $L^1(\Omega)$ is a metrizable spaces [43], the proof is complete. ☐

6.1.1 A level set algorithm for the inverse potential problem

We propose an explicit iterative algorithm derived from the optimality conditions (23) and (24) for the Tikhonov functional $\mathcal{G}_{\varepsilon,\alpha}$.

For the inverse potential problem with Dirichlet boundary condition ($\gamma_1 = 1$ and $\gamma_2 = 0$) the algorithm reads as:

Given σ and g;

1. `Evaluate the residual` $r_k := F_1(P_\varepsilon(\phi_k, \psi_k^1, \psi_k^2)) - y^\delta = (w_k)_\nu|_{\partial\Omega} - y^\delta$`, where` w_k `solves`
$$-\nabla \cdot (\sigma \nabla w_k) = P_\varepsilon(\phi_k, \psi_k^1, \psi_k^2)\,, \texttt{ in } \Omega\,; \qquad w_k = g\,, \texttt{ at } \partial\Omega\,.$$

2. `Evaluate` $h_k := F_1'(P_\varepsilon(\phi_k, \psi_k^1, \psi_k^2))^*(r_k) \in L^2(\Omega)$`, solving`
$$\Delta h_k = 0\,, \texttt{ in } \Omega\,; \qquad h_k = r_k\,, \texttt{ at } \partial\Omega\,.$$

3. `Calculate` $\delta\phi_k := L_{\varepsilon,\alpha,\beta}(\phi_k, \psi_k^1, \psi_k^2)$ `and` $\delta\psi_k^j := L^j_{\varepsilon,\alpha,\beta}(\phi_k, \psi_k^1, \psi_k^2)$`, as in (24).`

4. `Update the level set function` ϕ_k `and the level values` ψ_k^j`,` $j = 1, 2$`:`
$$\phi_{k+1} = \phi_k + \tfrac{1}{\alpha}\,\delta\phi_k\,, \qquad \psi_{k+1}^j = \psi_k^j + \tfrac{1}{\alpha}\,\delta\psi_k^j\,.$$

Table 1: Iterative algorithm based on the level set approach for the inverse potential problem.

Each step of this iterative method consists of three parts (see Table 1): 1 - The residual $r_k \in L^2(\partial\Omega)$ of the iterate (ϕ_k, ψ_k^j) is evaluated (this requires solving one elliptic BVP of Dirichlet type); 2 - The L^2–solution h_k of the adjoint problem for the residual is evaluated (this corresponds to solving one elliptic BVP of Dirichlet type); 3 - The update $\delta\phi_k$ for the level-set function and the updates $\delta\psi_k^j$ for the level values are evaluated (this corresponds to multiplying two functions).

In [38], a level set method was proposed, where the iteration is based on an inexact Newton type method. The inner iteration is implemented using the conjugate gradient method. Moreover, the regularization parameter $\alpha > 0$ is kept fixed. In contrast to [38], in Table 1, we define $\delta t = 1/\alpha$ (as a time increment) in order to derive an evolution equation for the levelset function. Therefore, we are looking for a fixed point equation related to the system of optimality conditions for the Tikhonov functional. Moreover, the iteration is based on a gradient type method as in [13].

6.2 The Inverse Problem in Nonlinear Electromagnetism

Many interesting physical problems are model by quasi-linear elliptic equations. Examples of applications include the identification of inhomogeneity inside nonlinear magnetic materials form indirect or local measurements. Electromagnetic non-destructive tests aim to localize cracks or inhomogeneities in the steel production process, where impurities can be described by a piecewise smooth function, [8, 9, 5, 11].

In this section, we assume that $D \subset\subset \Omega$ is measurable and

$$u = \begin{cases} \psi_1\,, x \in D\,, \\ \psi_2\,, x \in \Omega\ D\,, \end{cases} \tag{27}$$

where $\psi_1, \psi_2 \in \mathbb{B}$ and $m > 0$.

The forward problem consists of solving the Poisson boundary value problem

$$-\nabla\cdot(u\nabla w) \;=\; f\,,\ \text{in}\ \Omega\,,\quad w \;=\; g\ \text{on}\ \partial\Omega\,, \tag{28}$$

where $\Omega\subset\mathbb{R}^n$ with $\partial\Omega$ Lipschitz, the source $f\in H^{-1}(\Omega)$ and boundary condition $g\in H^{1/2}(\partial\Omega)$. It is well known that there exists a unique solution $w\in H^1(\Omega)$ such that $w-g\in H^1_0(\Omega)$ for the PDE (28), [12].

Assuming that during the production process the workpiece is contaminated by impurities and that such impurities are described by piecewise smooth function, the inverse electromagnetic problem consist in the identification and the localization of the inhomogeneities as well as the function values of the impurities. The localization of support and the tabulation of the inhomogeneities values can indicate possible sources of contamination in the magnetic material.

In other words, the inverse problem in electromagnetism consists in the identification of the support (shape) and the function values of ψ^1,ψ^2, of the coefficient function $u(x)$ defined in (27). The voltage potential g is chosen such that its corresponding the current measurement $h:=(w)_\nu|_{\partial\Omega}$, available as a set of continuous measurement in $\partial\Omega$. This problem is known in the literature as the inverse problem for the Dirichlet-to-Neumann map [24].

With this framework, the problem can be modeled by the operator equation

$$\begin{aligned} F_2\,:\,D(F)\subset L^1(\Omega)&\to H^{1/2}(\partial\Omega)\\ u&\longmapsto F_2(u):=w|_{\partial\Omega}\,, \end{aligned} \tag{29}$$

where the potential profile $g=w|_{\partial\Omega}\in H^{1/2}(\Omega)$ is given.

The authors in [11] investigated a level set approach for solve an inverse problems of identification of inhomogeneities inside a nonlinear material, from local measurements of the magnetic induction. The assumption in [11] is that part of the inhomogeneities are given by a crack localized inside the workpiece and that outside the crack region, magnetic conductivities are nonlinear and they depends on the magnetic induction. In other words, that $\psi_1=\mu_1$ and $\psi_2=\mu_2(|\nabla w|^2)$, where μ_1 is the (constant) air conductivity and $\mu_2=\mu_2(|\nabla w|^2)$ is a nonlinear conductivity of the workpiece material, whose values are assumed be known. In [11], they also present a successful iterative algorithm and numerical experiment. However, in [11], the measurements and therefore the data are given in the whole Ω. Such amount of measurements are not reasonable in applications. Moreover, the proposed level set algorithm is based on an optimality condition of a least square functional with $H^1(\Omega)$-semi-norm regularization. Therefore, there is no guarantee of existence of minimum for the proposed functional.

Remark 4. *Note that $F_2(u)=T_D(w)$, where T_D is the Dirichlet trace operator. Moreover, since $T_D:H^1(\Omega)\to H^{1/2}(\partial\Omega)$ is linear and continuous [12], we have $\|T_D(w)\|_{H^{1/2}(\partial\Omega)}\le c\|w\|_{H^1(\Omega)}$.*

In the following lemma, we prove that the operator F_2 satisfies the Assumption **(A2)**.

Lemma 16. *Let the operator $F_2\,:\,D(F)\subset L^1(\Omega)\longrightarrow H^{1/2}(\partial\Omega)$ as defined in (29). Then, F_2 is continuous with the respect to the $L^1(\Omega)$ topology.*

Proof. Let $u_n,u_0\in D(F)$ and w_n,w_0 denoting the respective solution of (25). The linearity of equation (28) implies that $w_n-w_0\in H^1_0(\Omega)$ and it satisfies

$$\nabla\cdot(u_n\nabla w_n)-\nabla\cdot(u_0\nabla w_0)=0\,, \tag{30}$$

with homogeneous boundary condition. Therefore, using the weak formulation for (30) we have

$$\int_\Omega \left(\nabla \cdot (u_n \nabla w_n) - \nabla \cdot (u_0 \nabla w_0) \right) \varphi dx = 0 \,, \quad \forall \varphi \in H^1_0(\Omega) \,.$$

In particular, the weak formulation holds true for $\varphi = w_n - w_0$. From the Green formula [12] and the assumption that $m > 0$ (that guarantee elipticity of (28)), follows that

$$m \|\nabla w_n - \nabla w_0\|^2_{L^2(\Omega)} \leq \int_\Omega u_n |\nabla w_n - \nabla w_0|^2 dx \leq \int_\Omega |(u_n - u_0)||\nabla w_0||(\nabla w_n - \nabla w_0)| dx \,. \quad (31)$$

From [28, Theorem 1], there exist $\varepsilon > 0$ (small enough) such that $w_0 \in W^{1,p}(\Omega)$ for $p = 2 + \varepsilon$. Using the Hölder inequality [12] with $1/p + 1/q = 1/2$ (note that $q > 2$ in the equation (31), follows that

$$m \|\nabla w_n - \nabla w_0\|^2_{L^2(\Omega)} \leq \|u_n - u_0\|_{L^q(\Omega)} \|\nabla w_0\|_{L^p(\Omega)} \|\nabla w_n - \nabla w_0\|_{L^2(\Omega)} \,. \quad (32)$$

Therefore, using the Poincaré inequality [12] and equation (32), we have

$$\|w_n - w_0\|_{H^1(\Omega)} \leq C \|u_n - u_0\|_{L^q(\Omega)} \,,$$

where the constant C depends only of $m, \Omega, \|\nabla w_0\|$ and the Poincaré constant. Now, the assertion follows from Lemma 2 and Remark 4. □

6.2.1 A level set algorithm for inverse problem in nonlinear electromagnetism

We propose an explicit iterative algorithm derived from the optimality conditions (23) and (24) for the Tikhonov functional $\mathcal{G}_{\varepsilon,\alpha}$. Each iteration of this algorithm consists in the following steps: In the first step the residual vector $r \in L^2(\partial\Omega)$ corresponding to the iterate $(\phi_n, \psi^1_n, \psi^2_n)$ is evaluated. This requires the solution of one elliptic BVP's of Dirichlet type. In the second step the solutions $v \in H^1(\Omega)$ of the adjoint problems for the residual components r are evaluated. This corresponds to solving one elliptic BVP of Neumann type and to computing the inner-product $\nabla w \cdot \nabla v$ in $L^2(\Omega)$. Next, the computation of $L_{\varepsilon,\alpha,\beta}(\phi_n, \psi^1_n, \psi^2_n)$ and $L^j_{\varepsilon,\alpha,\beta}(\phi_n, \psi^1_n, \psi^2_n)$ as in (24). The four step is the updates of the level-set function $\delta\phi_n \in H^1(\Omega)$ and the level function values $\delta\psi^j_n \in \mathtt{BV}(\Omega)$ by solve (23).

The algorithm is summarized in Table 2.

7 Conclusions and Future Directions

In this article, we generalize the results of convergence and stability of the level set regularization approach proposed in [14, 13], where the level values of discontinuities are not piecewise constant inside of each region. We analyze the particular case, where the set Ω is divide in two regions. However, it is easy to extend the analysis for the case of multiple regions adapting the multiple level set approach in [15, 14].

We apply the level set framework for two problems: the inverse potential problem and in an inverse problem in nonlinear electromagnetism with piecewise non-constant solution. In both case, we prove that the parameter-to-solution map satisfies the Assumption **(A1)**. The inverse potential problem application is a natural generalization of the problem computed in [15, 13, 14]. We also investigate the applicability of an inverse problem in nonlinear electromagnetism in the

1. Evaluate the residual $r := F_2(P_\varepsilon(\phi_n, \psi_n^1, \psi_n^2)) - y^\delta = w|_{\partial\Omega} - g^\delta$, where $w \in H^1(\Omega)$ solves
$$\nabla\cdot(P_\varepsilon(\phi_n, \psi_n^1, \psi_n^2)\,\nabla w) = f\,, \text{ in } \Omega\,; \qquad w = g\,, \text{ at } \partial\Omega\,.$$
2. Evaluate $F_2'(P_\varepsilon(\phi_n, \psi_n^1, \psi_n^2))^*\, r := \nabla w \cdot \nabla v \in L^2(\Omega)$, where w is the function computed in Step 1. and $v \in H^1(\Omega)$ solves
$$\nabla\cdot(P_\varepsilon(\phi_n, \psi_n^1, \psi_n^2)\,\nabla v) = 0 \text{ in } \Omega\,; \qquad v_\nu = r\,, \text{ at } \partial\Omega\,.$$
3. Calculate $L_{\varepsilon,\alpha,\beta}(\phi_n, \psi_n^1, \psi_n^2)$ and $L^j_{\varepsilon,\alpha,\beta}(\phi_n, \psi_n^1, \psi_n^2)$ as in (24).
4. Evaluate the updates $\delta\phi \in H^1(\Omega)$, $\delta\psi^j \in \mathrm{BV}(\Omega)$ by solving (23)
5. Update the level set functions $\phi_{n+1} = \phi_n + \frac{1}{\alpha}\delta\phi$, and the level function values $\psi^j_{n+1} = \psi^j_n + \frac{1}{\alpha}\delta\psi^j$.

Table 2: An explicit algorithm based on the proposed level set iterative regularization method.

identification of inhomogeneities inside a nonlinear magnetic workpiece. Moreover, we propose iterative algorithm based on the optimality condition of the smooth Tikhonov functional $\mathcal{G}_{\varepsilon,\alpha}$.

A natural continuation of this paper is the numerical implementation. Level set numerical implementations for the inverse potential problem was done before in [15, 14, 13], where the level values are assumed to be constant. Implementations of level set methods for resistivity/conductivity problem in elliptic equation have been intensively implemented recently e.g., [9, 18, 31, 41, 37, 11, 5].

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