The exposure buildup factor formulation in a slab and rectangle geometry by the LTS_N method

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Abstract: In this paper, we report a formulation for the exposure buildup factor by solving the one-dimensional photon transport equation in a heterogeneous slab by the LTS_N method, assuming the Klein-Nishina scattering kernel as the scattering differential cross-section as well as the multigroup model in the wavelength variable. We present numerical simulations and comparisons with available results in the literature for a multilayered slab composed of water, iron and lead. We also report an analytical solution to the exposure buildup factor by solving the photon transport equation in a rectangle applying the LTS_N nodal method.

Keywords: exposure buildup factor; Klein-Nishina scattering kernel; multigroup model; LTS_N nodal method.

Reference to this paper should be made as follows: Amaral Rodriguez, B.D., de Vilhena, M.T., Borges, V. and Malamut, C. (2007) 'The exposure buildup factor formulation in a slab and rectangle geometry by the LTS_N method', *Int. J. Low Radiation*, Vol. 4, No. 2, pp.102–116.

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1 Introduction

In the last decade, the LTS_N method, which solves, analytically, the discrete ordinates equation (S_N equation) in a slab by the Laplace transform technique, has made an appearance in related literature. The main idea consists of the following steps: application of the Laplace transform technique to the set of S_N equations, solution of the resulting algebraic equation by the matrix diagonalisation approach and inversion of the transformed angular flux by standard results of the Laplace transform theory. Here, analytical solution means that no approximation is made along the solution derivation. This methodology has been applied to a broad class of transport and radiative transfer problems. Some situations in which this methodology appears are the following: a general analytical approach to the one-group, one-dimensional transport equation by Barichello and Vilhena (1993); determination of the criticality parameters in heterogeneous slabs by the LTS_N method by Borges and Derivi (2001); the LTS_N method: a new analytical approach to solve the neutron transport equation by Vilhena and Barichello (1991); an analytical solution to the multigroup slab geometry discrete ordinates problems by Vilhena and Barichello (1995); extension of the LTS_N formulation for discrete ordinates problems without azimuthal symmetry by Segatto and Vilhena (1994); a new iterative method to solve the radiative transfer equation by Vilhena and Segatto (1996); the LTS_N solution for radiative transfer problems without azimuthal symmetry with severe anisotropy by Brancher *et al*. (1999); analytical solution of the discrete ordinates problem by the decomposition method by Vargas and Vilhena (1997); a closed-form solution for the one-dimensional radiative conductive problem by the decomposition and LTS_N methods by Vargas and Vilhena (1998); a closed-form solution to one-dimensional linear and non-linear radiative transfer problems by Vilhena and Barichello (1999); inverse problems for estimating bottom boundary conditions of natural waters in engineering by Velho *et al*. (2003); determining source term and boundary conditions in hydrological optics by Retamoso *et al*. (2001); estimation of boundary conditions in hydrologic optics by Retamoso *et al*. (2002); determination of the effective multiplication factor in a slab by the LTS_N method by Batistela *et al*. (1999); criticality by the LTS_N method by Batistela *et al*. (1997); recent advances in the LTS_N method for criticality calculations in slab geometry by Orengo *et al*. (2004); the LTS_N solution to the neutron transport equation in spherical geometry by Vasques *et al*. (2003); particle transport in the 1-D diffusive atomic mix limit by Larsen *et al*. (2005); and the convergence of the LTS_N method was proved by Pazos and Vilhena (1999; 2000). On the other hand, recently, the LTS_N method has been applied to the solution of the multidimensional S_N nodal equations in cartesian geometry by Hauser (2002), Pazos *et al*. (2003) and Zabadal *et al*. (1995), and in the convex domain by Zabadal *et al*. (1997), considering one-group energy and isotropic scattering. To our knowledge, this methodology has not yet been applied to the solution of the transport equation assuming the Klein-Nishina scattering kernel and multigroup model for the wavelength variable.

Therefore, in the first part of this work, we step forward by solving the transport equation in a slab considering the Klein-Nishina scattering kernel and multigroup model by the LTS_N method. Bearing in mind the analyticity and the mentioned proved convergence of the LTS_N method, we are confident to emphasise that we can generate benchmark results in the exposure buildup factor by this methodology, controlling the accuracy by increasing N. In the second part, we report a two-dimensional LTS_N solution for a homogeneous rectangle assuming the Klein-Nishina scattering kernel and multigroup model. The main idea relies on the solution of the two one-dimensional S_N equations resulting from transverse integration of the S_N equations in the rectangle by the LTS_N method, considering the leakage angular fluxes approximated exponentially, which allow us to determine a closed-form solution for the exposure buildup factor. Despite the lack of numerical validation, we are confident to affirm that the reported solution is actually a solution to the considered problem because the convergence of the LTS_N nodal solution discussed by Hauser *et al*. (2005) has been proven. Indeed, to reach our objectives, we organised the paper as follows: In Section 2, we present the LTS_N analytical solution to the exposure buildup factor in a slab assuming Klein-Nishina scattering kernel and multigroup model. In Section 3, we report numerical simulations and comparisons with available results in the literature. In Section 4, we display the two-dimensional LTS$_N$ nodal solution for the exposure buildup factor considering the Klein-Nishina scattering kernel and multigroup model. Finally, in Section 5, we present a discussion about the methodology considered.

2 The LTS$_N$ solution

In order to determine an analytical solution to the exposure buildup factor in a slab by the LTS$_N$ method, let us consider the following S$_N$ problem:

$$\mu_n \frac{\partial}{\partial x} I_{jn}(x) + \mu_{lj} I_{jn}(x) = \frac{\Delta}{3} \sum_{l=0}^{L} \frac{2l+1}{2} \times \\ \times \sum_{r=1}^{G} c_r \alpha k_{rj} P_l(1+\lambda_r - \lambda_j) P_l(\mu_n) \sum_{i=1}^{N} P_l(\mu_i) I_{ri}(x) \omega_i, \tag{1}$$

subject to vacuum boundary conditions, for $j = 1 : G$, $n = 1 : N$,

where:

G = the number of energy groups (wavelengths)
N = the Gaussian quadrature's order
μ_n = the roots of Legendre polynomial, ordered in decreasing fashion: $-1 < \mu_N < \ldots < \mu_{\frac{N}{2}+1} < 0 < \mu_{\frac{N}{2}} < \ldots < \mu_1 < 1$,
ω_i = the weights of Gaussian quadrature
μ_{lj} = the linear attenuation coefficient,
$I_{jn}(x) = I(x, \lambda_j, \mu_n)$ = the angular flux at μ_n direction for the j-th group
$k_{rj} = k(\lambda_r, \lambda_j)$ = Klein-Nishina scattering kernel, defined as follows:

$$k(\lambda_r, \lambda_j) = \frac{3}{8} \frac{\lambda_r}{\lambda_j} \left(\frac{\lambda_r}{\lambda_j} + \frac{\lambda_j}{\lambda_r} - \sin^2\theta \right). \tag{2}$$

Following the idea of the LTS$_N$ method, we begin applying the Laplace transform technique to Equation (1), with the resulting linear algebraic system:

$$s\overline{I_{jn}(s)} + \frac{\mu_{lj}}{\mu_n} \overline{I_{jn}(s)} - \frac{\Delta}{3\mu_n} \sum_{l=0}^{L} \frac{2l+1}{2} \sum_{r=1}^{G} c_r \alpha k_{rj} P_l(1+\lambda_r - \lambda_j) P_l(\mu_n) \\ \times \sum_{i=1}^{N} P_l(\mu_i) \omega_i \overline{I_{ri}(s)} = I_{jn}(0), \tag{3}$$

for $j = 1 : G$, $n = 1 : N$, which can be recast in matrix form, as shown below:

$$A_{jn}(s)\overline{I_{jn}(s)} = I_{jn}(0) + \overline{Z_{j-1}(s)}. \tag{4}$$

Here, $\overline{I_{jn}(s)}$ is the N component of the angular flux Laplace transformed vector and $I_{jn}(0)$ is the N component of the angular flux vector at $x = 0$. They have the form

$$\overline{I_{jn}(s)} = [\overline{I_{j1}(s)}\overline{I_{j2}(s)} \ldots \overline{I_{jN}(s)}]^T, \tag{5}$$

$$I_{jn}(0) = [I_{j1}(0) I_{j2}(0) \ldots I_{jN}(0)]^T . \tag{6}$$

On the other hand, the entries of the $(N \times N)$ matrix $A_{jn}(s)$ are written as follows:

$$a_{pq} = \begin{cases} s + \dfrac{\mu_{tj}}{\mu_p} - \dfrac{\Delta}{3\mu_p} \sum_{l=0}^{L} \dfrac{2l+1}{2} c_j \alpha k_{jj} P_l(\mu_p) P_l(\mu_p) \omega_q & \text{se } p = q \\ -\dfrac{\Delta}{3\mu_p} \sum_{l=0}^{L} \dfrac{2l+1}{2} c_j \alpha k_{jj} P_l(\mu_p) P_l(\mu_q) \omega_q & \text{se } p \neq q \end{cases} \tag{7}$$

and the scattering term reads

$$\overline{Z_{j-1}(s)} = \sum_{i=1}^{j-1} H_i \overline{I_{in}(s)}, \tag{8}$$

where the components of constant matrix H_i are given by the following:

$$h_{pq} = \begin{cases} \dfrac{\Delta}{3\mu_p} \sum_{l=0}^{L} \dfrac{2l+1}{2} c_i \alpha k_{ij} P_l(1 + \lambda_i - \lambda_j) P_l(\mu_p) P_l(\mu_p) \omega_q & \text{se } p = q \\ -\dfrac{\Delta}{3\mu_p} \sum_{l=0}^{L} \dfrac{2l+1}{2} c_i \alpha k_{ij} P_l(1 + \lambda_i - \lambda_j) P_l(\mu_p) P_l(\mu_q) \omega_q & \text{se } p \neq q. \end{cases} \tag{9}$$

Bearing in mind that for the Klein-Nishina scattering kernel the wavelength ranges from λ_0 to $\lambda_0 + 2$ (λ_0 is the wavelength of the slab incoming radiation), we discretise, without loss of generality, this interval in five sub-intervals, meaning five groups, with the main feature that the first group (Group 1) corresponds to the sub-interval with the shortest wavelength and higher energy and Group 5, to the sub-interval with the longest wavelength and lowest energy.

Solving Equation (4) recursively for increasing wavelength (j from 1 to 5), due to the down-scattering, we come out with the result:

$$\overline{I_{jn}(s)} = [A_j(s)]^{-1} I_{jn}(0) + [A_j(s)]^{-1} \overline{Z_{j-1}(s)}. \tag{10}$$

Making the Laplace inversion of the above ansatz, we have

$$I_{jn}(x) = \mathcal{L}^{-1}\{[A_{jn}(s)]^{-1} I_{jn}(0)\} + \mathcal{L}^{-1}\{[A_{jn}(s)]^{-1} \overline{Z_{j-1}(s)}\}, \tag{11}$$

which by the Heaviside expansion technique can be recast as follows:

$$I_{jn}(x) = \sum_{k=1}^{jn} \beta_k e^{s_k x} I_{jn}(0) + Z_{j-1}(x) * \mathcal{L}^{-1}\{[A_{jn}(s)]^{-1}\}, \tag{12}$$

where:

$$\beta_k = \left. \frac{Adj(A_{jn}(s))}{\dfrac{d}{ds}[detA_{jn}(s)]} \right|_{s=s_k} \tag{13}$$

and the values of s_k are the roots of the characteristics polynomial of the $A_{jn}(s)$ matrix. Here, a star denotes convolution. The exponential character of the generic method combined with the fact that the s_k parameters increase in magnitude with N implies that this formulation, in the proposed form, is not appropriate to solving large thickness transport problem. Fortunately, this difficulty was suppressed by introducing the ensuing modification for the basis space solution:

$$I_{jn}(x) = \left(\sum_{k=1_{s_k>0}}^{jn} \beta_k e^{-s_k(a-x)} + \sum_{k=1_{s_k<0}}^{jn} \beta_k e^{s_k x} \right) I^*_{jn}(0) + Z_{j-1}(x) * \mathcal{L}^{-1}\{[A_j(s)]^{-1}\}, \tag{14}$$

where $I^*_{jn}(0) = C(a) \times I_{jn}(0)$ is the N component modifying column vector and $C(a)$ reads as follows:

$$C(a) = \left(\sum_{k=1_{s_k>0}}^{jn} \beta_k e^{-s_k a} + \sum_{k=1_{s_k<0}}^{jn} \beta_k \right)^{-1}. \tag{14a}$$

We determine the new arbitrary constant $I^*_{jn}(0)$ applying the boundary conditions.

The generalisation of the LTS$_N$ solution for a heterogeneous slab assumes the Klein-Nishina scattering kernel and multigroup model is done in a straightforward manner. Indeed, we apply the LTS$_N$ solution to each sublayer and evaluate the integration constants applying the boundary and interface conditions. This procedure leads to the following result for an arbitrary slab in the domain depicted in Figure 1:

$$I^r_{jn}(x) = \left(\sum_{k=1_{s^r_k>0}}^{jn} \beta^r_k e^{-s^r_k[(x_r - x_{r-1})-x]} + \sum_{k=1_{s^r_k<0}}^{jn} \beta^r_k e^{s^r_k x} \right) I^{r\,*}_{jn}(0) + Z^r_{j-1}(x) * \mathcal{L}^{-1}\{[A^r_j(s)]^{-1}\}, \tag{15}$$

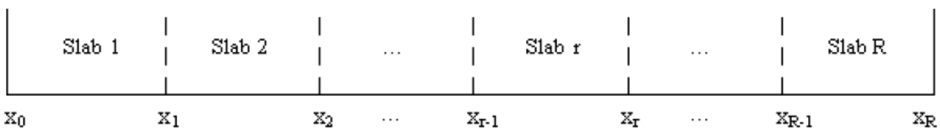
where $0 < x < x_R - x_{R-1}$, $r = 1 : R$ and

$$I^r_{jn}(x) = [I^r_{11}(x) \ldots I^r_{1N}(x) \ldots I^r_{j1}(x) \ldots I^r_{jN}(x)]^T, \tag{16}$$

$$I^r_{jn}(0) = [I^r_{11}(0) \ldots I^r_{1N}(0) \ldots I^r_{j1}(0) \ldots I^r_{jN}(s)]^T. \tag{17}$$

Here $I^r_{jn}(x)$ is the angular flux for the generic slab r.

Figure 1 Multilayered domain



3 Numerical results for the one-dimensional problem

In order to illustrate the aptness of the discussed methodology to solve transport problems in a slab considering the Klein-Nishina scattering kernel and multigroup model, in the sequel we report numerical simulations for the discussed exposure buildup factor. To reach this goal, we evaluate the exposure buildup factor defined according to Fitzgerald *et al*. (1967) as the sum of the product of the attenuation coefficient of the air with the scalar flux for all radiations, including the incident flux, divided by the attenuation coefficient of the air for the incident flux multiplied by the incident scalar flux, shown as follows:

$$B_e(x) = \frac{\sum_{i=0}^{j} \mu_l^{air}(\lambda_i)\varphi_i(x,\lambda_i)}{\mu_l^{air}(\lambda_0)\varphi_0(x,\lambda_0)}. \tag{18}$$

Here, the subscript index 0 indicates the incident flux, $\mu_l^{air}(\lambda_i)$ is the attenuation coefficient of the air for the wavelength λ_i, $\mu_l^{air}(\lambda_0)$ is the attenuation coefficient of the air for the incident flux (for wavelength λ_0), $\varphi_i(x, \lambda_i)$ is the scalar flux for the wavelength λ_i and $\varphi_0(x, \lambda_0)$ is the incident scalar flux. In what follows, we present numerical results for three problems.

Problem 1 Let us consider a multilayered slab with two regions, composed of water (μ_{lj} = 0.0707 cm^2/g, mfp = 1.0) and lead (μ_{lj} = 0.06848 cm^2/g, mfp = 4.0, 5.0, 10.0, 20.0, 30.0 and 40.0) and under vacuum boundary conditions.

In Table 1 we present the LTS$_N$ numerical simulations for the exposure buildup factor and comparisons with the ones (EGS$_4$ results) attained by Hirayama and Shin (1998). Bearing in mind that EGS$_4$ results are generated for the one-group model, given a closer look to the results in Table 1, we promptly realise a good coincidence. To underline the proved convergence of the LTS$_N$ method in Table 2, we display the numerical convergence of the LTS$_N$ results for increasing N. In fact, observing the results for N = 14 and N = 16, we notice a coincidence of six significant digits. Recalling the character of the solution, in the sense that no approximation is made along its derivation, except for the round-off error, this means that we may affirm that $B_e(x)$ = 2.30 and $B_e(x)$ = 3.57 are the exact results for Problem 1 and, consequently, are benchmark results for Problem 1. We bolster this affirmative recalling the proved convergence of the LTS$_N$ method.

Table 1 Numerical exposure buildup factor simulations in water and lead composition

	Water 1.0 mfp + Lead	
Mfp	*LTS$_{16}$*	*EGS$_4$*
4.0	2.30	2.31
5.0	2.07	2.08
10.0	3.57	3.59
20.0	5.29	5.31
30.0	6.77	6.79
40.0	8.26	8.27

Problem 2 To check the influence of the attenuation coefficient on the exposure buildup factor solution, let us consider the two-layered slab composed of water (μ_{lj} = 0.0707 cm^2/g, mfp = 1.0) and iron (μ_{lj} = 0.0596 cm^2/g, mfp = 4.0, 5.0, 10.0, 20.0, 30.0 and 40.0) and under vacuum boundary conditions.

Table 2 LTS_N numerical convergence

N	*4 mfp*	*10 mfp*
2	2.29043291	3.56575823
4	2.29124721	3.56773931
6	2.29593785	3.56981458
8	2.29921456	3.57019857
12	2.30013541	3.57022344
14	2.30014775	3.57022455
16	2.30014786	3.57022457

Table 3 Numerical exposure buildup factor simulations in water and iron composition

	Water 1.0 mfp + Iron	
mfp	LTS_{16}	EGS_4
4.0	4.99	5.01
5.0	6.21	6.23
10.0	13.9	13.9
20.0	36.3	36.3
30.0	67.6	67.5
40.0	101	101

Problem 3 Let us consider a heterogeneous slab with two regions, composed of lead (μ_{lj} = 0.06848 cm^2/g, mfp = 1.0) and iron (μ_{lj} = 0.0596 cm^2/g, mfp = 4.0, 5.0, 10.0, 20.0, 30.0 and 40.0) and under vacuum boundary conditions.

Table 4 Numerical exposure buildup factor simulations in lead and iron composition

	Lead 1.0 mfp + Iron	
mfp	LTS_{16}	EGS_4
4.0	4.87	4.86
5.0	6.34	6.28
10.0	15.4	15.3
20.0	41.5	41.4
30.0	78.4	78.3
40.0	118	117

From the analysis of the results encountered for the above problems, we promptly realise a good agreement between the LTS_{16} and EGS_4 results. Bearing in mind the previously mentioned LTS_{16} results accuracy in Problem 1, we can also emphasise, supported by the same arguments, that the LTS_{16} results encountered for the exposure buildup factor are also benchmark results. We must also mention that we have done all the calculations in an AMD Athlon 1700 (1.4 GHz) microcomputer. Furthermore, the maximum computational time observed to generate all the results in each table is 90 s.

4 The LTS$_N$ nodal solution in a rectangle

Let us consider the two-dimensional S$_N$ nodal problem assuming the Klein-Nishina scattering kernel and multigroup model:

$$\begin{aligned}&\mu_n \frac{\partial}{\partial x} I_{jn}(x,y) + \eta_n \frac{\partial}{\partial y} I_{jn}(x,y) + \mu_{lj} I_{jn}(x,y) = \\ &= \frac{\Delta}{3} \sum_{l=0}^{L} \frac{2l+1}{2} \sum_{r=1}^{G} c_r \alpha k_{rj} P_l(1+\lambda_r - \lambda_j) P_l(\mu_n) \sum_{i=1}^{N} P_l(\mu_i) I_{ri}(x,y) \omega_i,\end{aligned} \tag{19}$$

subject to vacuum boundary conditions in a rectangle $0 \le x \le a$ and $0 \le y \le b$. Here $j = 1 : G$, $n = 1 : N$, $N = \dfrac{M(M+2)}{2}$ is the cardinality of the discrete ordinates set (number of discrete directions), M represents the order of the angular quadrature, G is the number of energy groups (wavelengths), μ_{lj} is the linear attenuation coefficient, $I_{jn}(x, y) = I(x, y, \lambda_j, \Omega_n)$ is the angular flux at the discrete direction $\Omega_n = (\mu_n, \eta_n)$ for the j-th group, the values of ω_i are the Lewis and Miller (1984) quadrature weights and $k_{rj} = k(\lambda_r, \lambda_j)$ is the Klein-Nishina scattering kernel defined by Equation (2).

To construct the LTS$_N$ nodal solution for Problem (19), we begin performing the transverse integration of this equation. This procedure yields to the set of the ensuing two coupled S$_N$ equations:

$$\begin{aligned}&\eta_n \frac{d}{dy} I_{jny}(y) + \frac{\mu_n}{a}[I_{jn}(a,y) - I_{jn}(0,y)] + \mu_{lj} I_{jny}(y) = \\ &= \frac{\Delta}{3} \sum_{l=0}^{L} \frac{2l+1}{2} \sum_{r=1}^{G} c_r \alpha k_{rj} P_l(1+\lambda_r - \lambda_j) P_l(\mu_n) \sum_{i=1}^{N} P_l(\mu_i) I_{riy}(y) \omega_i,\end{aligned} \tag{20}$$

for $j = 1 : G$, $n = 1 : N$. Here $I_{jn}(a, y)$ and $I_{jn}(0, y)$ are the angular fluxes exiting at the boundary and the average angular flux is written as follows:

$$I_{jny}(y) = \frac{1}{a} \int_0^a I_{jn}(x,y)dx. \tag{21}$$

$$\begin{aligned}&\mu_n \frac{d}{dx} I_{jnx}(x) + \frac{\eta_n}{b}[I_{jn}(x,0) - I_{jn}(x,b)] + \mu_{lj} I_{jnx}(x) = \\ &= \frac{\Delta}{3} \sum_{l=0}^{L} \frac{2l+1}{2} \sum_{r=1}^{G} c_r \alpha k_{rj} P_l(1+\lambda_r - \lambda_j) P_l(\mu_n) \sum_{i=1}^{N} P_l(\mu_i) I_{rix}(x) \omega_i,\end{aligned} \tag{22}$$

for $j = 1 : G$, $n = 1 : N$. Here $I_{jn}(x, b)$ and $I_{jn}(x, 0)$ are the angular fluxes exiting at the boundary and the average angular flux is written as follows:

$$I_{jnx}(x) = \frac{1}{b} \int_0^b I_{jn}(x,y)dy. \tag{23}$$

At this point we are in a position to apply the LTS$_N$ method. Indeed, we begin applying the Laplace transform technique in Equation (20). This procedure yields the following:

$$\begin{aligned}&s\overline{I_{jny}(s)}+\frac{\mu_{lj}}{\eta_n}\overline{I_{jny}(s)}-\frac{\Delta}{3\eta_n}\sum_{l=0}^{L}\frac{2l+1}{2}\sum_{r=1}^{G}c_r\alpha k_{rj}P_l(1+\lambda_r-\lambda_j)P_l(\mu_n)\times\\&\times\sum_{i=1}^{N}P_l(\mu_i)\overline{I_{riy}(s)}\omega_i=I_{jny}(0)-\frac{\mu_n}{a\eta_n}[\overline{I_{jn}(a,s)}-\overline{I_{jn}(0,s)}],\end{aligned}\tag{24}$$

for $j = 1 : G$ and $n = 1 : N$, which can be recast in matrix form as follows:

$$(sI-B_{jny})\overline{I_{jny}(s)}=I_{jny}(0)+\overline{Z_{(j-1)y}(s)}+\overline{S_{jny}(s)}.\tag{25}$$

Here $\overline{I_{jny}(s)}$ is the N component of the angular flux Laplace transformed vector in the y variable and $I_{jny}(0)$ is the N component of the angular flux vector in the y variable at $y = 0$. They have the form

$$\overline{I_{jny}(s)}=[\overline{I_{j1y}(s)}\overline{I_{j2y}(s)}\ldots\overline{I_{jNy}(s)}]^T,\tag{26}$$

$$\overline{I_{jny}(0)}=[I_{j1y}(0)I_{j2y}(0)\ldots I_{jNy}(0)]^T.\tag{27}$$

On the other hand, the components of matrix B_{jny} are given by

$$b_y(p,q)=\begin{cases}-\dfrac{\mu_{lj}}{\eta_p}+\dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L}\dfrac{2l+1}{2}c_j\alpha k_{jj}P_l(\mu_p)P_l(\mu_p)\omega_q & \text{se } p=q\\ \dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L}\dfrac{2l+1}{2}c_j\alpha k_{jj}P_l(\mu_p)P_l(\mu_q)\omega_q & \text{se } p\neq q\end{cases}\tag{28}$$

and the scattering term reads

$$\overline{Z_{(j-1)y}(s)}=\sum_{i=1}^{j-1}H_{iy}\overline{I_{iny}(s)},\tag{29}$$

where the entries of constant matrix H_{iy} are written as follows:

$$h_y(p,q)=\begin{cases}\dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L}\dfrac{2l+1}{2}c_i\alpha k_{ij}P_l(1+\lambda_i-\lambda_j)P_l(\mu_p)P_l(\mu_p)\omega_q & \text{se } p=q\\ -\dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L}\dfrac{2l+1}{2}c_i\alpha k_{ij}P_l(1+\lambda_i-\lambda_j)P_l(\mu_p)P_l(\mu_q)\omega_q & \text{se } p\neq q.\end{cases}\tag{30}$$

The vector $\overline{S_{jny}(s)}$ has the generic component

$$S_{jiy}(s)=\frac{\mu_i}{a\eta_i}[\overline{I_{ji}(a,s)}-\overline{I_{ji}(0,s)}].\tag{31}$$

A similar procedure in the x variable leads to the ensuing linear algebraic system:

$$\begin{aligned}&s\overline{I_{jnx}(s)}+\frac{\mu_{lj}}{\mu_n}\overline{I_{jnx}(s)}-\frac{\Delta}{3\mu_n}\sum_{l=0}^{L}\frac{2l+1}{2}\sum_{r=1}^{G}c_r\alpha k_{rj}P_l(1+\lambda_r-\lambda_j)P_l(\mu_n)\times\\&\times\sum_{i=1}^{N}P_l(\mu_i)\overline{I_{rix}(s)}\omega_i=I_{jnx}(0)-\frac{\eta_n}{b\mu_n}[\overline{I_{jn}(s,b)}-\overline{I_{jn}(s,0)}],\end{aligned}\tag{32}$$

which again can be recast in the matrix form as follows:

$$(sI - A_{jnx})\overline{I_{jnx}(s)} = I_{jnx}(0) + \overline{Z_{(j-1)x}(s)} + \overline{S_{jnx}(s)}. \tag{33}$$

Here $\overline{I_{jnx}(s)}$ is the N component of the angular flux Laplace transformed vector in the x variable and $I_{jnx}(0)$ is the N components of the angular flux vector in the x variable at $x = 0$. They have the form

$$\overline{I_{jnx}(s)} = [\overline{I_{j1x}(s)}\overline{I_{j2x}(s)}\ldots\overline{I_{jNx}(s)}]^T, \tag{34}$$

$$I_{jnx}(0) = [I_{j1x}(0)I_{j2x}(0)\ldots I_{jNx}(0)]^T. \tag{35}$$

On the other hand, the entries of matrix A_{jnx} are written as follows:

$$a_x(p,q) = \begin{cases} -\dfrac{\mu_{lj}}{\mu_p} + \dfrac{\Delta}{3\eta_p}\sum_{l=0}^{L}\dfrac{2l+1}{2}c_i\alpha k_{jj}P_l(\mu_p)P_l(\mu_p)\omega_q & \text{se } p = q \\ \dfrac{\Delta}{3\mu_p}\sum_{l=0}^{L}\dfrac{2l+1}{2}c_j\alpha k_{jj}P_l(\mu_p)P_l(\mu_q)\omega_q & \text{se } p \neq q. \end{cases} \tag{36}$$

and the scattering term reads

$$\overline{Z_{(j-1)x}(s)} = \sum_{i=1}^{j-1} H_{ix}\overline{I_{inx}(s)}, \tag{37}$$

where the constant matrix H_{ix} have elements given by

$$h_x(p,q) = \begin{cases} \dfrac{\Delta}{3\mu_p}\sum_{l=0}^{L}\dfrac{2l+1}{2}c_i\alpha k_{ij}P_l(1+\lambda_i-\lambda_j)P_l(\mu_p)P_l(\mu_p)\omega_q & \text{se } p = q \\ -\dfrac{\Delta}{3\mu_p}\sum_{l=0}^{L}\dfrac{2l+1}{2}c_i\alpha k_{ij}P_l(1+\lambda_i-\lambda_j)P_l(\mu_p)P_l(\mu_q)\omega_q & \text{se } p \neq q, \end{cases} \tag{38}$$

and the vector $\overline{S_{jnx}(s)}$ reads

$$\overline{S_{jix}(s)} = \frac{\eta_i}{b\mu_i}[\overline{I_{ji}(s,b)} - \overline{I_{ji}(s,0)}]. \tag{39}$$

The LTS$_N$ solution for Equations (25) and (33) are given by the following:

$$\overline{I_{jny}(s)} = (sI - B_{jny})^{-1}[I_{jny}(0) + \overline{Z_{(j-1)y}(s)} + \overline{S_{jny}(s)}] \tag{40}$$

and

$$\overline{I_{jnx}(s)} = (sI - A_{jnx})^{-1}[I_{jnx}(0) + \overline{Z_{(j-1)x}(s)} + \overline{S_{jnx}(s)}]. \tag{41}$$

Taking the Laplace inversion of the above ansatz we get,

$$I_{jny}(y) = \mathcal{L}^{-1}\{(sI - B_{jny})^{-1}[I_{jny}(0) + \overline{Z_{(j-1)y}(s)} + \overline{S_{jny}(s)}]\} \tag{42}$$

and

$$I_{jnx}(x) = \mathcal{L}^{-1}\{(sI - A_{jnx})^{-1}[I_{jnx}(0) + \overline{Z_{(j-1)x}(s)} + \overline{S_{jnx}(s)}]\}. \tag{43}$$

Here, we do not proceed further to evaluate the Laplace inversion of Equations (42) and (43) because the inversion of these solutions has the same expression as the one in Equation (14). The explanation for this affirmative comes from the fact that the matrices A_{jnx} and B_{jny} are also nondegenerate. To complete the solution, we have to determine the unknown leakage angular fluxes at boundary, namely, $I_{jn}(x, 0)$, $I_{jn}(0, y)$, $I_{jn}(x, b)$ and $I_{jn}(a, y)$. Following the work of Hauser (2002), which states that the exponential approximation gives the best results for the two-dimensional LTS$_N$ nodal solution for deep penetration problems, we assume the ensuing approximation for the leakage angular fluxes:

$$I_{jn}(x,0) = F_{jn} e^{-sign(\mu_n)\Lambda x} \tag{44}$$

$$I_{jn}(0,y) = G_{jn} e^{-sign(\eta_n)\Lambda y} \tag{45}$$

$$I_{jn}(x,b) = O_{jn} e^{-sign(\mu_n)\Lambda x} \tag{46}$$

$$I_{jn}(a,y) = P_{jn} e^{-sign(\eta_n)\Lambda y} \tag{47}$$

where $sign(\mu)$ denotes the signal function:

$$sign(\mu) = \begin{cases} 1 \; if \;\; \mu > 0 \\ -1 \; if \;\; \mu > 0 \end{cases} \tag{48}$$

and Λ represents the decay constant parameter, which has to be chosen a priori. In this work, we assume Λ, as Hauser (2002) did, as being the absorption cross-section given by:

$$\Lambda = \sigma_a = \sigma_t - \sigma_s. \tag{49}$$

The functions $sign(\mu_n)$ and $sign(\eta_n)$, which appear in Equations (44)–(47), guarantee that the approximated angular fluxes will decay for any discrete direction. Replacing (44)–(47) in Equations (42) and (43), the x- and y-averaged angular fluxes solutions are complete after the Laplace transform inversion. Applying the boundary conditions, we determine the integration constants and consequently the two-dimensional LTS$_N$ nodal solution is well determined. Once the averaged angular fluxes are known, the closed-form solution for the two-dimensional exposure buildup factor is given by Equation (18), done by just replacing $\varphi(x, \lambda)$ with $\varphi(x, y, \lambda)$.

5 Conclusion

Concluding, we would like to point out our confidence that we hit our objective in this work because we succeeded in extending the LTS$_N$ solution for problems in cartesian geometry, requiring the Klein-Nishina scattering kernel and multigroup model. This procedure allows us to derive a closed-form solution to the exposure buildup factor. To this point, we must recall that, before this work, the LTS$_N$ solution was restricted to transport problems demanding isotropic and anisotropic scattering differential cross-section. Indeed, beginning our final analysis by looking at the first part of this work, we must emphasise that the LTS_N solution reported keeps the analytical feature, in the sense that no approximation is made along its derivation from the S_N equations, except for the round-off error. Furthermore, the proved error-bound estimates and convergence by Pazos and Vilhena (1999; 2000), Hauser *et al*. (2005) assure that the LTS_N solution converges to the exact solution when N goes to infinity. We must also underline that the LTS_N method is quite general in the sense that it can now be applied to handle problems that demand the Klein-Nishina scattering kernel, which satisfies the error bound and convergence requirements. Therefore, we are emphasising that the mathematical analysis of the LTS_N is complete, regarding the issues of solution construction, error-bound estimates, convergence and results validation. Consequently, this method is a quite robust approach under either the mathematical or computational point of view to generate benchmark results. We reinforce this affirmative bearing in mind that, besides the analytical character of the solution, the LTS_N method solves transport problems that requires large N ($N \leq 2000$) with a small computational time. On the other hand, concerning the second part, we begin by saying that the major claim refers to the issue of mathematical analysis that consists of the solution derivation presented in Section 2, as well as the proved error-bound estimates and convergence by Pazos and Vilhena (1999; 2000) and Hauser *et al*. (2005). The justificative for this affirmative comes from the fact that the solution to the two-dimensional S_N nodal problem is reduced to the solution of a set of two one-dimensional S_N equations, which are quite similar to the ones validated in Section 1. Therefore, we believe that the mathematical analysis, somehow, compensates for the lack of results validation for the two-dimensional LTS_N nodal solution. Now, regarding the topic of analyticity, we must emphasise that the unique approximation made along the derivation of the LTS_N nodal solution was in the leakage angular flux at boundary. In addition, we find it relevant to comment that our analysis is restricted to the homogeneous rectangle, because we attain, in a straightforward manner, the solution for the heterogeneous rectangle proceeding likewise in the heterogeneous slab. Finally, pursuing our objective of searching for analytical solutions, we focus our future attention on the issue of extending the LTS_N nodal solution for the three-dimensional problem in a heterogeneous parallelepiped assuming the discussed kernels. We also intend to complete the mathematical analysis concerning the issues of error-bound estimates and convergence. We hope to show by this procedure the aptness and robustness of the LTS_N method to solve a broad class of transport problems.

Acknowledgements

The first and second authors are gratefully indebted to CNPq (Conselho Nacional de Desenvolvimento Científíco e Tecnológico) for the partial financial support to this work.

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